Al-Qadisiyah Journal of Pure Science

Volume 25 | Number 1 Article 8

1-7-2020

Weak Topology On Modular Space

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Available online at http://qu.edu.iq/journalsc/index.php/JOPS

Al-Qadisiyah Journal Of Pure Science (QJPS)

ISSN 2411-3514 ONLINE
ISSN 1997-2490 PRINTED

Vol. 25, No. 1, Math pp. 46 –49, Year2020

Weak topology on modular space $\sigma(X_M, X^*)$

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Received : 18/1/2020 Accepted :29/2/2020

ABSTRACT:

The purpose of this investigation is to define the weak topology on a specific topological spaces (modular spaces) this topology generated by the modular dual space X^* i.e. it is generated by class of mappings $\mathcal{F} = \{f_\alpha : X \to Y_\alpha : f_\alpha \text{ linear and continuous}\}$ when $\mathcal{F} = X^*$, we denote it $\sigma(X_M, X^*)$ and induced characterization to the structure of this topology. Afterwards, we proof it is hausdorff space finally we discuss the property of convergence in $\sigma(X_M, X^*)$.

KEYWORDS: modular space , weak topology , weak topology on modular

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1. Introduction:

In preliminaries we talk about the modular spaces and the weak topology (initial topology). The notion of modular functions appears in 1950 by H. Nakano [15] who referred to a class of functions from any vector space $\boldsymbol{X}$ over a field $\boldsymbol{F}$ (where $\boldsymbol{F} = \mathbb{R}\ \boldsymbol{or}\ \mathbb{C}$) into the interval $[0, \infty]$ (i.e. $M: X \to [0, \infty]$) with conditions (i) $M(x) = 0 \Leftrightarrow x = 0$, (ii) $M(\lambda x) = M(x), \forall \lambda \in F$ with $|\lambda| = 1$, (iii) $M(\lambda_1 x + \lambda_2 y) \leq M(\lambda_2 x) + M(\lambda_2 y)\ \forall\ x, y \in X$ and $\forall \lambda, \lambda_1, \lambda_2\ in\ F$.

The subspace $X_M = \{x \in X: M(\lambda x) \to 0, whenever\ \lambda \to 0\}$ of a vector space X is called modular space see [6] and it metric space when define the distance between any two points x, y in X_M by $D_M(x, y) = M(x - y)$ [4-8] that's mean X_M has the topology generated by D_M. In the part two of preliminaries we introduced the concept of the weak topology in general see coherent topology also [10-18]. After that we investigate the structuration of the weak topology endowed with X_M when $\mathcal{F} = X^*$ the family of linear continuous functional [11].

2. Preliminaries

We divided this section into two part. Lets begin with

2.1. Modular space

In this part, we present some definitions and characterizations concerns the notion of modular spaces

Definition 2.1.1: [8] Let X a linear space over afield $\mathbb{F}$. A function $M: X \to [0, \infty]$ is called modular if:

1. $M(x) = 0 \Leftrightarrow x = 0$
2. $M(\lambda x) = M(x)$ for $\lambda \in F$ with $|\lambda| = 1$.
3. $M(\lambda_1 x + \lambda_2 y) \leq M(\lambda_1 x) + M(\lambda_2 y)$ iff $\lambda_1, \lambda_2 \geq 0$, $x \in X$.

The space X_M given by $X_M = \{ x \in X : M(\lambda x) \to 0$ whenever $\lambda \to 0 \}$.

If (3) in definition modular space replaced by $M(\lambda_1 x + \lambda_2 y) \leq \lambda_1 M(x) + \lambda_2 M(y)$, for all $\lambda_1, \lambda_2 \geq 0, \lambda_1 + \lambda_2 = 1$, for all $x, y \in M$, the modular M is called convex modular. **[15]**

By condition (3) above , if $y = 0$ then $M(\alpha x) = M\left(\frac{\lambda_1}{\lambda_2} \lambda_2 x\right) \leq M(\lambda_2 x)$, for all λ_1, λ_2 in $F, 0 < \lambda_1 < \lambda_2$. Thus M is increasing function .

Remark 2.1.2:

1. A modular space X_M is a metric space with $D_M(x, y) = M(x - y)$, for all $x, y \in X$. **See [4-6]**
2. Any modular space is a topological linear space, Moreover ,it is Hausdorff space **[14]**

Now, For the definition of topological linear space Definition (2,1.3) : [1-2-13]

i- The M-open ball $B_r(x)$ centered $x \in X_M$ with radius $r > 0$ as

$$B_r(x) = \{y \in X_M: M(y - x) < r\}.$$

ii- The M-closed ball $\overline{B_r(x)}$centered$x \in X_M$ with radius $r > 0$ as

$$\overline{B_r(x)} = \{y \in X_M: M(y - x) \leq r\}.$$

iii- The class of all M-balls in a modular space X_M generates a topology which makes X_MHausdorff topological linear space.

iv- Every M-ball is convex set, since every modular space is locally convex Hausdorff topological linear space.

v- Let (X, M) be a modular space and $A \subseteq X$ we say that A is open set if for every $x \in A$ there exist $r > 0, \ni B_r(x) \subset A$.

vi- A subset A of X is said to be closed if its complement is open, that is , $A^c = X - A$ is open.

2.2. The weak topology

In this part we introduced the original notion of initial topology and a property of convergence in this topology

Definition 2.2.1: [9] Let X be any nonempty set and let $\{(X_\alpha, \tau_\alpha): \alpha \in \Delta\}$ be any arbitrary nonempty collection of topological space. For each $\in \Delta$, let f_α be a function of X into X_α . Then the topology τ on X generated by the collection $\mathcal{G} = \{f_\alpha^{-1}(A): A \in \tau_\alpha, \alpha \in \Delta\}$ is called the weak topology on X determined by the collection $\{f_\alpha: \in \Delta\}$. $\mathcal{G}$ is called the defining subbase of τ and the collection β of all finite intersections of members of $\mathcal{G}$ is called the defining of τ.

Remark 2.2.2: Let X a nonempty set and let $\{(X_\alpha, \tau_\alpha): \alpha \in \Delta\}$ be a nonempty family of topological spaces indexed by Δ. The weak topology generated by the family of functions $\mathcal{F} = \{f_\alpha: X \to X_\alpha, \alpha \in \Delta\}$ is the topology generated by the sub-basis open sets $\mathcal{G} = \{f_\alpha^{-1}(A_\alpha): A_\alpha \in \tau_\alpha,\ \alpha \in \Delta\}$. We denote the topology generated by $\mathcal{F}$ on X by $\sigma(X, \mathcal{F})$.

Definition 2.2.3: [11] A set G in X is said to be weakly open or open in a topology $\sigma(X, \mathcal{F})$ if for all $y \in G$, there exists a finite subset ρ of Δ and open sets $\{A_\alpha\}_{\alpha \in \rho}$ such that $A_\alpha \subseteq X_\alpha$ for all $\alpha \in \rho$,$y \in \bigcap_{i=1}^{n} f_i^{-1}(A_i)$ which mean that $\forall i \in \rho$, $f_i(y) \in A_i$.

Definition 2.2.4: [10] A sequence $\{x_n\}$ in X is said to be weakly convergent to $x \in X$ if it is converges in the topology $\sigma(X, \mathcal{F})$, and we write that $x_n \overset{w}{\to} x$.

That is $x_n \overset{w}{\to} x$ if and only if $f_\alpha(x_n) \to f_\alpha(x)$ for all $\alpha \in \Delta$.

3. The mean result

In this section, $\boldsymbol{X_M}$ is modular space on the field $\boldsymbol{F}$ where, $\boldsymbol{F} = \mathbb{R}$ or $\mathbf{F} = \mathbb{C}$

we do not assume that it is X complete ,

Let $X_\alpha = F$ and $f_\alpha: X \longrightarrow X_\alpha$ be a function . Now let $\mathcal{F} = \{f_\alpha: \alpha \in \Delta\}$, and let $\mathcal{G} = \{\Gamma \subseteq \Delta: \Gamma \text{ finite}\}$.

We have the weak topology on X_M generated by $\mathcal{F}$ has the basis

$$\beta = \left\{ \bigcap_{\alpha \in \Gamma} [\![f_\alpha^{-1}(-\varepsilon, \varepsilon): \Gamma \in \mathcal{G}, \varepsilon > 0 \}]\!] \right.$$

thus a set G is weakly open in X_M iff given G, there exists $\alpha_1.\alpha_2, \ldots, \alpha_n \in \Delta$ such that $x \in \bigcap_{i=1}^{n} f_{\alpha_i}^{-1}(-\varepsilon, \varepsilon) \subseteq G$ this is gives $|f_{\alpha_i}(x)| < \varepsilon$ for $i = 1, 2, \ldots, n$

A sub basis open set containing a point $x_0 \in X_M$ is of the form

$$f_\alpha^{-1}(f_\alpha(x_0) - \varepsilon, f_\alpha(x) - \varepsilon)$$

for each $\alpha \in \Lambda$ and for each $\varepsilon > 0$. Hence it can be of the form

$\beta(y_0; f_{.1}, f_2, \ldots, f_n; \varepsilon) = \{ x \in X: |f_\alpha(x) - f_\alpha(y_0)| < \varepsilon\}$ for $f_i \in \mathcal{F}$, $i = 1, 2, \ldots, n\ n \geq 1$, $\varepsilon > 0$.

Remark 3.6:

1. $y_0 \in \beta(y_{01}; f_1, f_{.2}, \ldots, f_n, \varepsilon)$
2. Let $h_i \in X^*\ i = 1, 2, \ldots, n$.Let$\beta_1 =$

$\beta(y_0; f_1, f_2, \dots, f_n; \varepsilon_1)$,$\beta_2 = \beta(y_0; h_1, h_2, \dots, h_n; \varepsilon_2)$ and $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Then $\beta(y_0; f_1, f_{\cdot 2}, \dots, f_n, g_1, g_2, \dots, g_n; \varepsilon) \subseteq \beta_1 \cap \beta_2$

3. If $x_1 \in \beta(y_0; f_1, f_2, \dots, f_n; \varepsilon)$ then there exists a $\delta > 0$ such that $\beta(y_0; f_1, f_2, \dots, f_n; \delta) \subseteq \beta(y_0; f_1, f_2, \dots, f_n; \varepsilon)$
4. Let $x_0, y_0 \in X$ with $y_0 \neq x_0$ and $\varepsilon_1, \varepsilon_2 > 0$ .Let $x \in \beta(x_0; f_1, f_2, \dots, f_n; \varepsilon_1) \cap \beta(y_0; g_1, g_2, \dots, g_n; \varepsilon_2)$ .Then there is $\delta > 0$ such that $\beta(x_0; f_1, f_2, \dots, f_n, g_1, g_2, \dots, g_n; \delta) \subseteq \beta_1 \cap \beta_2$
5. Let $z_0, y_0 \in X$ with $z_0 \neq y_0$. suppose that there exists $f \in \mathcal{F}$ such that $f(z_0) \neq f(y_0)$. Then $\beta(z_0, f, \frac{\varepsilon}{2}) \cap \beta\left(y_0, f, \frac{\varepsilon}{2}\right) = \phi$

Remark 3.7: Let X_M be modular space over a field F and let $\mathcal{F} = X^*$ the dual space of X i.e $X^* = \{f: X \longrightarrow F$, f is continuous linear functional$\}$, then the weak topology generated by $\mathcal{F}$ on X_M denoted by $\sigma(X_M, X^*)$

Definition 3.8: The weak topology on modular space X_M is the topology $\sigma(X, \{f\}_{f \in X^*})$. For convenience, it is denoted $\sigma(X, X^*)$ or $\sigma(X, \mathcal{F})$ .In other words, the weak topology on modular space X_M is (X_M, X^*) , which generated by members of X^*.

Note that a sub biasc of open set about $y_0 \in X_M$ of the form

$U(y_0, f, \varepsilon) = \{x \in X: |f(x) - f(y_0)| < \varepsilon\}$ for all $f \in X^*, \varepsilon > 0$ and a basic open set about $y_0 \in X_M$ in the topology $\sigma(X_M, X^*)$ is of the form

$\beta(y_0; f_1, f_2, \dots, f_n; \varepsilon) = \{x \in X: |f_i(x) - f_i(y_0)| < \varepsilon\}$ for all choices of $f_i \in X^*, i = 1,2, \dots, n, n \geq 1$, $\varepsilon > 0$

A set A in modular space X_M is said to be weakly open if for all $\in A$, there exists $f_1, f_2, \dots, f_n \in X^*, n \geq 1$ such that for all $\varepsilon > 0$,

$A_y = \{x \in X: |f_i(x) - f_i(y)| < \varepsilon, i = 1,2, \dots, n$

In the following important result, we sufficient condition for Hausdorff topology.

Theorem 3.9: If X_M modular space, Then the topology $\sigma(X_M, X^*)$is Hausdorff.

Proof:

Let $x, y \in X_M$ such that $\neq y$.

Since $x \neq y$, then $x - y \neq 0$

$\Longrightarrow M(x - y) > 0$,$\Longrightarrow$ there exists $\varepsilon > o$, such that $y \notin \beta_\varepsilon(x)$

Since $\beta_\varepsilon(x)$ convex open set , we know that it can be strictly separated from $\{y\}$. by the Hahn-banach theorem there exists $f \in X^*$ and $\lambda \in \mathbb{R}$ such that $f(s) < \lambda < f(y)$ for all $s \in \beta_\varepsilon(x)$

In this case $f(x) < \lambda < f(y)$. therefore $x \in f^{-1}(-\infty, \lambda)$ and $y \in f^{-1}(\lambda, \infty)$.

Since $f^{-1}(-\infty, \lambda)$ and $y \in f^{-1}(\lambda, \infty)$ are preimages of open subsets of $\mathbb{R}$ by a linear functional, then those two sets are weakly open , and they are disjoint.

Hence the space $\sigma(X_M, X^*)$ is Hausdorff because there is two disjoint weakly open sets for all x , y in X_M such that every set contained one of x or y.

In the next, we present the main results related to the weak convergent in modular space X_M.

Theorem 3.10: In a modular space X_M with the topology $\sigma(X_M, X^*)$

1. The topology $\sigma(X_M, X^*)$ is weaker than the modular topology of X.
2. In a modular space X_M . If $x_n \overset{w}{\rightarrow} x$, then x is unique.
3. In a modular space X_M . If $x_n \overset{w}{\rightarrow} x$, then every subsequence of $\{x_n\}$ converges weakly to x.
4. In a modular space X_M . If $x_n \overset{w}{\rightarrow} x$, then the sequence $\{M(x_n)\}$ is bounded
5. If $\{x_n\}$ a sequence in a modular space X_M and $x \in X$, then $x_n \overset{w}{\rightarrow} x$ iff $f(x_n) \longrightarrow f(x)$ for all $f \in X^*$
6. In a modular space X_M . A strongly converging sequence converges weakly
7. If $\{x_n\}$ a sequence in a modular space X_M s.t $x_n \overset{w}{\rightarrow} x$ and let $\{f_n\}$ be a sequence in X^* such that $f_n \rightarrow f$, then $f_n(x_n) \rightarrow x_n(x)$

Proof:

1. Let X be a modular space and let G be a weakly open in . To prove G is open in X. Let $x \in G$ then there exists $f_1, f_2, \dots, f_n \in X^*$, $n \geq 1$ such that for all $\varepsilon > 0$, $G_x = \{y \in X: |f_i(x) - f_i(y)| < \varepsilon i = 1,2, \dots, n\} \subseteq G$.
   Since $f_i \in X^*$, $i = 1,2, \dots, n$, then f_i is continuous linear functional on X for all $i = 1,2, \dots, n$.
2. Suppose that $x_n \overset{w}{\rightarrow} y$ to prove that $x = y$ and let $f \in X^*$ since $x_n \overset{w}{\rightarrow} x$ and $x_n \overset{w}{\rightarrow} y$, then $f(x_n) \rightarrow x$ and $f(x_n) \rightarrow y$. Since the limit point is unique , then $x = y$
3. Since $\{f(x_n)\}$ convergent sequence in F for all $f \in X^*$

So that every subsequence of $\{f(x_n)\}$ converges and has the same limit as a sequence.

4. Since $x_n \overset{w}{\rightarrow} x$, then $\{f(x_n)\}$ is convergent sequence in F for all $f \in X^*$

Thus $\{f(x_n)\}$ is bounded

Then there exists $M_f > o$ such that $|f(x_n)| \leq M_f$ for all , where M_f constant depending of f but not n

Using the canonical function $\varphi: X \rightarrow X^*$, we can define $g_n \in X^{**}$ by

$g_n(f) = f(x_n)$ for all $f \in X^*$

Then for all , $|g_n(f)| = |f(x_n)| \leq M_g$

That is, the sequence $\{|g_n(f)|\}$ is bounded for every $f \in X^*$ and since X^* is complete , then $\{M(g_n)\}$ is bounded

Now since $M(g_n) = M(x_n)$, then $\{M(x_n)\}$ is bounded

5. Suppose $x_n \overset{w}{\rightarrow} x$ and $\{f_n\}$ sequence in X^*

Take $f \in X^*$, and $\varepsilon > 0$.

Then $D = \{y \in X: |f(y) - f(x)| < \varepsilon\}$ is open set in $\sigma(X_M, X^*)$ and D containing x.

Put $k \in \mathbb{N}$ s.t $x_n \in D$ for all $n \geq k$

Thus for all weakly open set D there exists $k \in \mathbb{N}$ such that $x_n \in D$ for all $n \geq k$ and that $\forall f \in X^*$

$\Longrightarrow f(x_n) \rightarrow f(x)$

Conversely , assume that $f(x_n) \rightarrow f(x), \forall \in X^*$

Let D be open in X_M containing .

Chooce a $\lambda > 0$ and $f_j \in X^*$, $j = 1,2, \dots, m$ such that $\{y \in X: |f_j(x) \rightarrow f_j(y)| < \varepsilon, j = 1,2, \dots, m\} \subseteq D$. As $f_j(x_n) \rightarrow f_j(x)\ \forall\ j = 1,2, \dots, m$

There exists $k_j \in \mathbb{N}$ such that $|f_j(x_n) - f_j(x)| < \varepsilon$ for all $\geq k_j$.

Let $k_0 = \max\{k_1, k_2, \dots, k_m\}$

Then for $\geq k_0$, we have $|f_j(x_n) - f_j(x)| < \varepsilon$ for all

$n \geq k_0$

Hence $x_n \overset{w}{\rightarrow} x$

6. If $\{x_n\}$ a sequence in a modular space X_M s.t $x_n \rightarrow x$

Since $x_n \rightarrow x$, then $M(x_n - x) \rightarrow 0 \quad as \quad n \rightarrow \infty$. Let $f \in X^*$ that is f continuous linear functional $|f(x_n) - f(x)| = |f(x_n - x)| \rightarrow 0 \quad as \quad n \rightarrow \infty$. Since f arbitrary , then $x_n \overset{w}{\rightarrow} x$

7. If $\{x_n\}$ a sequence in modular space X_M such that $x_n \overset{w}{\rightarrow} x$, and let $\{f_n\}$ be a sequence in X^*. Since $x_n \overset{w}{\rightarrow} x$, then $f(x_n) \rightarrow f(x)$, for all $f \in X^*$ .Thus $f_n(x_n) \rightarrow f_n(x)$

Theorem 3.11: Let X_M be modular space

i. If $x_n \overset{w}{\rightarrow} x$ and $y_n \overset{w}{\rightarrow} y$, then $x_n + y_n \overset{w}{\rightarrow} x + y$

ii. If $x_n \overset{w}{\rightarrow} x$ and $\lambda \in F$, then $\lambda x_n \overset{w}{\rightarrow} \lambda x$

Proof:

i. Since $x_n \overset{w}{\rightarrow} x$, then $f(x_n) \rightarrow f(x) \ \forall f \in X^*$

And since $y_n \overset{w}{\rightarrow} y$, then $f(y_n) \rightarrow f(y) \ \forall f \in X^*$

$|f(x_n + y_n) - f(x + y)| = |f(x_n) + f(y_n) - f(x) - f(y)| \leq |f(x_n) - f(x)| + |f(y_n) - f(y)|$

Since $f(x_n) \rightarrow f(x)$ and $(y_n) \rightarrow f(y)$, then $|f(x_n) - f(x)| \rightarrow 0$ and $|f(y_n) - f(y)| \rightarrow 0$

Hence $|f(x_n + y_n) - f(x + y)| \rightarrow 0$ for all $f \in X^*$

ii. Since $x_n \overset{w}{\rightarrow} x$, then $f(x_n) \rightarrow f(x)$, $\forall f \in X^*$

Thus $|f(\lambda x_n) - f(\lambda x)| = |\lambda f(x_n) - f(x)| = |\lambda||f(x_n) - f(x)| \rightarrow 0$, $\forall f \in X^*$

Hence $\lambda x_n \overset{w}{\rightarrow} \lambda x$.

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