Weak Topology On Modular Space

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Weak topology on modular space
\[ \sigma(X_M, X^*) \]

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ABSTRACT:
The purpose of this investigation is to define the weak topology on a specific topological spaces (modular spaces) this topology generated by the modular dual space \( X^* \) i.e. it is generated by class of mappings \( \mathcal{F} = \{ f_\alpha : X \rightarrow Y \ ; \ f_\alpha \text{ linear and continuous} \} \) when \( \mathcal{F} = X^* \), we denote it \( \sigma(X_M, X^*) \) and induced characterization to the structure of this topology. Afterwards, we proof it is Hausdorff space finally we discuss the property of convergence in \( \sigma(X_M, X^*) \).

KEYWORDS: modular space, weak topology, weak topology on modular
1. Introduction: In preliminaries we talk about the modular spaces and the weak topology (initial topology). The notion of modular functions appears in 1950 by H. Nakano [15] who referred to a class of functions from any vector space $X$ over a field $F$ (where $F = \mathbb{R}$ or $\mathbb{C}$) into the interval $[0, \infty]$ (i.e., $M : X \to [0, \infty]$) with conditions (i) $M(x) = 0 \iff x = 0$, (ii) $M(\lambda x) = |\lambda| M(x)$, for all $x \in X$ with $|\lambda| = 1$, (iii) $M(\lambda_1 x + \lambda_2 y) \leq M(\lambda_1 x) + M(\lambda_2 y)$ for all $x, y \in X$ and $\lambda_1, \lambda_2 \geq 0$. The subspace $X_M = \{ x \in X : M(\lambda x) \to 0 \}$, whenever $\lambda \to 0$ of a vector space $X$ is called modular space see [6] and it metric space when define the distance between any two points $x, y$ in $X_M$ by $D_M(x, y) = M(x - y)$ [4-8] that’s mean $X_M$ has the topology generated by $D_M$. In the part two of preliminaries we introduced the concept of the weak topology in general see coherently topology also [10-18]. After that we investigate the structurization of the weak topology endowed with $X_M$ when $\mathcal{F} = X^*$ the family of linear continuous functional [11].

2. Preliminaries

We divided this section into two part. Lets begin with

2.1. Modular space

In this part, we present some definitions and characterizations concerning the notion of modular spaces

Definition 2.1.1: [8] Let $X$ a linear space over a field $F$. A function $M : X \to [0, \infty]$ is called modular if:
1. $M(x) = 0 \iff x = 0$,
2. $M(\lambda x) = M(x)$ for $\lambda \in F$ with $|\lambda| = 1$, 
3. $M(\lambda_1 x + \lambda_2 y) \leq M(\lambda_1 x) + M(\lambda_2 y)$ if and only if $\lambda_1, \lambda_2 \geq 0$ and $\lambda_1, \lambda_2 \in F$. 

The space $X_M$ given by $X_M = \{ x \in X : M(\lambda x) \to 0 \}$, whenever $\lambda \to 0$.

If (3) in definition modular space replaced by $M(\lambda_1 x + \lambda_2 y) \leq \lambda_1 M(x) + \lambda_2 M(y)$, for all $\lambda_1, \lambda_2 \geq 0$, $\lambda_1, \lambda_2 \in F$, then the modular $M$ is called convex modular. [15]

By condition (3) above, if $y = 0$ then $M(\lambda x) = M(\lambda_2 x)$, for all $\lambda_1, \lambda_2 \in F, 0 < \lambda_1 < \lambda_2$. Thus $M$ is increasing function.

Remark 2.1.2: 1. A modular space $X_M$ is a metric space with $D_M(x, y) = M(x - y)$, for all $x, y \in X$. See [4-6]

2. Any modular space is a topological linear space. Moreover, it is Hausdorff space [14]

Now, for the definition of topological linear space Definition (2.1.3) : [1-2-13]

i- The $M$-open ball $B_r(x)$ centered in $X_M$ with radius $r > 0$ as $B_r(x) = \{ y \in X_M : M(y - x) < r \}$.

ii- The $M$-closed ball $\overline{B}_r(x)$ centered in $X_M$ with radius $r > 0$ as $\overline{B}_r(x) = \{ y \in X_M : M(y - x) \leq r \}$.

iii- The class of all $M$-balls in a modular space $X_M$ generates a topology which makes $X_M$ Hausdorff topological linear space.

iv- Every $M$-ball is convex set, since every modular space is locally convex Hausdorff topological linear space.

v- Let $(X, M)$ be a modular space and $A \subseteq X$ we say that $A$ is open set if for every $x \in A$ there exist $r > 0$, $B_r(x) \subseteq A$.

vi- A subset $A$ of $X$ is said to be closed if its complement is open, that is $A^c = X - A$ is open.

2.2. The weak topology

In this part we introduced the original notion of initial topology and a property of convergence in this topology

Definition 2.2.1: [9] Let $X$ be any nonempty set and let $(\{x_n, t_n\} : n \in \Delta)$ be any arbitrary nonempty collection of topological space. For each $n \in \Delta$, let $f_n$ be a function of $X$ into $X_n$. Then the topology $\tau$ on $X$ generated by the collection $\mathcal{G} = \{f_n^{-1}(A) : A \in \tau_n, n \in \Delta\}$ is called the weak topology on $X$ determined by the collection $\{f_n : n \in \Delta\}$. $\tau$ is called the defining subbase of $\tau$ and the collection $\beta$ of all finite intersections of members of $\mathcal{G}$ is called the defining of $\tau$.

Remark 2.2.2: Let $X$ a nonempty set and let $(\{x_n, t_n\} : n \in \Delta)$ be any arbitrary nonempty family of topological spaces indexed by $\Delta$. The weak topology generated by the family of functions $\mathcal{F} = \{f_n : X \to X_n, n \in \Delta\}$ is the topology generated by the sub-base collection $\mathcal{G} = \{f_n^{-1}(A) : A \in \tau_n, n \in \Delta\}$. We denote the topology generated by $\mathcal{F}$ on $X$ by $\sigma(X, F)$.

Definition 2.2.3: [11] A set $G$ in $X$ is said to be weakly open or open in a topology $\sigma(X, F)$ if for all $y \in G$, there exists a finite subset $\rho$ of $\Delta$ and open sets $\{A_n\}_{n \in \rho}$ such that $A_n \subseteq X_{\rho}$ for all $n \in \rho$, and $G = \bigcap_{n \in \rho} f_n^{-1}(A_n)$ which mean that $\forall n \in \rho, f_n(y) \in A_n$.

Definition 2.2.4: [10] A sequence $\{x_n\}$ in $X$ is said to be weakly convergent to $x \in X$ if it converges in the topology $\sigma(X, F)$, and we write $x_n \to x$.

That is $x_n \to x$ if and only if $f_n(x_n) \to f_n(x)$ for all $n \in \Delta$.

3. The mean result

In this section, $X_M$ is modular space on the field $F$ where, $F = \mathbb{R}$ or $F = \mathbb{C}$ we do not assume that it is $X$ complete.

Let $X_M = F$ and $f_{\alpha} : X \to X_{\alpha}$ be a function. Now let $\mathcal{F} = \{f_\alpha : \alpha \in \Delta\}$, and let $G = \{G \subseteq \Delta : G \text{ finite} \}$. We have the weak topology on $X_M$ generated by $\mathcal{F}$ has the basis

$$
\beta = \left\{ \bigcap_{n \in \rho} f_n^{-1}(-\varepsilon, \varepsilon) : \varepsilon > 0 \right\}
$$

such a set $G$ is weakly open in $X_M$ iff given $G$, there exists $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Delta$ such that $x \in \bigcap_{n \in \rho} f_n^{-1}(-\varepsilon, \varepsilon) \subseteq G$ this is gives $f_{\alpha_i}(-\varepsilon, \varepsilon)$ for $i = 1, 2, \ldots, n$.

A sub basis open set containing a point $x_0 \in X_M$ is of the form

$$
f_{\alpha_i}^{-1}(\varepsilon, \varepsilon) = f_{\alpha_i}(x_0) - \varepsilon
$$

for each $\alpha \in \Delta$ and for each $\varepsilon > 0$. Hence it can be of the form

$$
\beta(\varepsilon) = \{ x \in X : f_{\alpha_i}(x) - f_{\alpha_i}(y_0) < \varepsilon \}
$$

for $f_i \in F, i = 1, 2, \ldots, n, \varepsilon > 0$.

Remark 3.6: 1.

$y_0 \in \beta(\varepsilon)$

2. Let $\beta_i \in X, i = 1, 2, \ldots, n$ let $\varepsilon = \frac{1}{n}$.
\[ \beta(Y; f_1, f_2, ..., f_n; \varepsilon_1) \triangleq \beta(Y; h_1, h_2, ..., h_n; \varepsilon_2) \]

and \( \varepsilon = \min(\varepsilon_1, \varepsilon_2) \). Then

\[ \beta(Y; f_1, f_2, ..., f_n; \varepsilon) \subseteq \beta(Y; f_1, f_2, ..., f_n; \varepsilon) \cap \beta(Y; f_1, f_2, ..., f_n; \varepsilon) \]

3. If \( x \in \beta(Y; f_1, f_2, ..., f_n; \varepsilon) \) then there exists a \( \delta > 0 \) such that

\[ \beta(Y; f_1, f_2, ..., f_n; \delta) \subseteq \beta(Y; f_1, f_2, ..., f_n; \varepsilon) \]

4. Let \( x_0, y_0 \in X \) with \( y_0 \neq x_0 \) and \( \varepsilon_1, \varepsilon_2 > 0 \). Then

\[ \beta(x_0, f_1, f_2, ..., f_n; \varepsilon) \cap \beta(y_0, g_1, g_2, ..., g_m; \varepsilon) \]

5. Let \( x_0, y_0 \in X \) with \( x_0 \neq y_0 \). Suppose that there exists \( f \) such that \( f(x_0) \neq f(y_0) \).

Then \( \beta(x_0, f_1, f_2, ..., f_n) \cap \beta(y_0, f_1, f_2, ..., f_n) = \emptyset \).

**Remark 3.7:** Let \( X_M \) be modular space over a field \( F \) and \( X' \) be the dual space of \( X \) i.e. \( X' = \{ f; f \to F, f \text{ is continuous functional} \} \), then the weak topology generated by \( f \) on \( X_M \) is denoted by \( \sigma(X_M, X') \).

**Definition 3.8:** The weak topology on modular space \( X_M \) is the topology \( \sigma(X, \{ f \}) \). For convenience, it is denoted \( \sigma(X, X') \) or \( \sigma(X, F) \). In other words, the weak topology on modular space \( X_M \) is \( (X_M, X') \), which is generated by members of \( X' \).

**Theorem 3.9:** If \( X_M \) is modular space. Then the topology \( \sigma(X_M, X') \) is Hausdroff.

**Proof:**

1. Let \( x, y \in X_M \) such that \( x \neq y \).

Since \( x \neq y \), then \( y - x \neq 0 \).

\[ M(x - y) > 0 \implies \text{there exists } \varepsilon > 0 \text{ such that } \beta(x, \varepsilon) \]

Since \( \beta(x, \varepsilon) \) is convex open set, we know that it can be strictly separated from \( y \) by the Hahn-banach theorem there exists \( f \in X' \) and \( \lambda \in \mathbb{R} \) such that

\[ f(x) < \lambda < f(y) \text{ for all } x \in \beta(x, \varepsilon) \]

In this case \( f(x) < \lambda < f(y) \) therefore \( x \in f^{-1}(-\infty, \lambda) \) and \( y \in f^{-1}(\lambda, \infty) \).

Since \( f^{-1}(\lambda, \infty) \) and \( f^{-1}(\lambda, \infty) \) are preimages of open subsets of \( \mathbb{R} \) by a linear functional, then those two sets are weakly open, and they are disjoint.

Hence, the space \( (X_M, X') \) is Hausdorff because there exist two disjoint weakly open sets for all \( x, y \in X_M \) such that every set contained one of \( x \) or \( y \).

In the next, we present the main results related to the weak convergent in modular space \( X_M \).

**Theorem 3.10:** In a modular space \( X_M \) with the topology \( \sigma(X_M, X') \).

1. The topology \( \sigma(X_M, X') \) is weaker than the modular topology of \( X \).

2. In a modular space \( X_M \). If \( x_n \to x \), then \( x_n \) is unique.

3. In a modular space \( X_M \). If \( x_n \to x \), then every subsequence of \( \{x_n\} \) converges weakly to \( x \).

4. In a modular space \( X_M \). If \( x_n \to x \), then the sequence \( \{M(x_n)\} \) is bounded.

5. If \( \{x_n\} \) is a sequence in a modular space \( X_M \) and \( x \in X \), then \( x_n \to x \) iff \( f(x_n) \to f(x) \) for all \( f \in X' \).

6. In a modular space \( X_M \). A strongly converging sequence converges weakly.

7. If \( \{x_n\} \) is a sequence in a modular space \( X_M \) s.t

\[ x_n \to x \text{ and } f(x_n) \neq f(x) \text{ be a sequence in } X' \text{ such that } f(x_n) \to f(x) \text{ then } f(x_n) \to f(x) \text{ in } X' \text{ for all } f \in X' \text{ for all } \varepsilon > 0 \].

**Proof:**

1. Let \( X \) be a modular space and let \( G \) be a weakly open in . To prove \( G \) is open in \( X \). Let \( x \in \subset G \). Then there exists \( f_1, f_2, ..., f_n \in X' \), \( \gamma > 0 \) such that for all \( \varepsilon > 0 \).

\[ G_e = \{ y \in X; |f_i(x) - f_i(y)| < i, i = 1,2, ..., n \} \subseteq G \]

Since \( x \in X \), \( i = 1,2, ..., n \). then \( x \) is continuous linear functional on \( X \) for all \( i = 1,2, ..., n \).

2. Suppose that \( x_n \to y \) to prove that \( x = y \) and \( f \in X' \) such that \( x_n \to x \) \( y \to f \) \( (x_n) \to (x) \) \( y \to (y) \).

3. Since \( \{f(x_n)\} \) convergent sequence in \( F \) for all \( f \in X' \).

So that every subsequence of \( \{f(x_n)\} \) converges and has the same limit as a sequence.

4. Since \( x_n \to x \). Then \( \{f(x_n)\} \) is convergent sequence in \( F \) for all \( f \in X' \).

Thus \( \{f(x_n)\} \) is bounded

Then there exists \( M_f > 0 \) such that \( |f(x_n)| \leq M_f \) for all \( f \) constant depending of \( b \) but not \( n \).

Using the canonical function \( \psi: X \to X' \), we can define \( g_n \in X' \) by

\[ g_n(f) = f(x_n) \text{ for all } f \in X' \]

Then for all \( n, g_n(f) = |f(x_n)| \leq M_f \), \( M_f \) is bounded. Since \( (g_n(f)) \) is bounded for all \( f \in X' \) and since \( X' \) is complete , then \( \{g_n(f)\} \) is bounded.

Now since \( g_n(f) = M(x_n) \), then \( \{M(x_n)\} \) is bounded.

5. Suppose \( x_n \to x \) and \( f(x_n) \) sequence in \( X' \).

Take \( f \in X' \) such that \( \varepsilon > 0 \).

\[ D = \{y \in X; |f(y) - f(x)| < \varepsilon \} \text{ is open set in } \sigma(X_M, X') \text{ and } D \text{ containing } x \]

Put \( k \in \mathbb{N} \) s.t. \( x_n \to k \) for all \( n \geq k \)

Thus all weakly open set \( D \) exists \( k \in \mathbb{N} \) such that \( x_n \in D \) for all \( n \geq k \) and \( \forall f \in X' \).

\[ f(x_n) \to f(x) \]

Conversely, assume that \( f(x_n) \to f(x) \). Then \( x \in X' \).

Let \( D \) be open in \( X_M \) containing .

Choose a \( \lambda > 0 \) and \( f_j \in X' \cdot j = 1,2, ..., m \) such that

\[ \{ y \in X; |f_j(x) - f_j(y)| < \varepsilon, j = 1,2, ..., m \} \subseteq D \text{ as } f_j(x) \to f_j(x) \forall j = 1,2, ..., m \]

There exists \( k_j \in \mathbb{N} \) such that \( |f_j(x_n) - f_j(x)| < \varepsilon \) for all \( \geq k_j \).

Let \( k_0 = \max(k_1, k_2, ..., k_m) \)

Then for \( \geq k_0 \), we have \( |f_j(x_n) - f_j(x)| < \varepsilon \) for all
\[ n \geq k_0 \]

Hence \( x_n \to x \)

6. If \( \{x_n\} \) a sequence in a modular space \( X_M \) s.t \( x_n \to x \)

Since \( x_n \to x \) then \( M(x_n - x) \to 0 \) as \( n \to \infty \), Let \( f \in X^* \) that is \( f \) continuous linear functional \( |f(x_n) - f(x)| = |f(x_n - x)| \to 0 \) as \( n \to \infty \) Since \( f \) arbitrary, \( x_n \to x \)

7. If \( \{x_n\} \) a sequence in modular space \( X_M \) such that \( x_n \to x \) and \( \{f_n\} \) a sequence in \( X^* \). Since \( x_n \to x \) then \( f(x_n) \to f(x) \) for all \( f \in X^* \). Thus \( f_n(x_n) \to f_n(x) \)

**Theorem 3.11**: Let \( X_M \) be modular space \( \{w\}

i. If \( x_n \to x \) and \( y_n \to y \), then \( x_n + y_n \to x + y \)

ii. If \( x_n \to x \) and \( \lambda \in F \), then \( \lambda x_n \to \lambda x \)

**Proof**: 

i. Since \( x_n \to x \), then \( f(x_n) \to f(x) \) for all \( f \in X^* \)

And since \( y_n \to y \), then \( f(y_n) \to f(y) \)

\[ |f(x_n + y_n) - f(x + y)| = |f(x_n) + f(y_n) - f(x) - f(y)| \leq |f(x_n) - f(x)| + |f(y_n) - f(y)| \]

Since \( f(x_n) \to f(x) \) and \( y_n \to y \), then \( f(y_n) \to f(y) \) \( \Rightarrow |f(y_n) - f(y)| \to 0 \)

Hence \( f(x_n + y_n) \to f(x + y) \) for all \( f \in X^* \)

ii. Since \( x_n \to x \), then \( f(x_n) \to f(x) \) for all \( f \in X^* \)

Thus \( |f(\lambda x_n) - f(\lambda x)| = |\lambda f(x_n) - \lambda f(x)| = |\lambda||f(x_n) - f(x)| \to 0 \) for all \( f \in X^* \)

Hence \( \lambda x_n \to \lambda x \)

**References**


