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Differential Sandwich Results For Univalent Functions

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Abstract :

In the present paper, we obtain some subordination and superordination Results involving the integral operator T_α for certain normalized analytic functions in the open unit disk. These results are applied to obtain sandwich results.

Keywords : Analytic function, differential subordination ,superordination ,sandwich sheorem, dominant, subordinant, integral operator.

2019 Mathematics Subject Classification: 30C45, 30C50.

1. Introduction:

Let H = H (U) be the class of analytic functions in the open unit disk U = {z ∈ ℂ : |z|< 1 }. For n a positive integer and a ∈ ℂ, let H [a,n] be the subclass of the function f ∈ H consisting of functions of the form :

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots \quad (a \in \mathbb{C}, n \in \mathbb{N}).$$

Also, let A be the subclass of H consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let f, g ∈ H. The function f is said to be subordinate to g, or g is said to be superordinate to f, if there exists a Schwarz function w analytic in U with w(0) = 0 and | w(z) | < 1 (z ∈ U), such that $f(z) = g(w(z))$, in such a case, we write $f \prec g$ or $f(z) \prec g(z)$ (z ∈ U). If g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let p, h ∈ H and $\psi(r, \delta, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$. If p and $\psi(p(z), zp'(z), z^2p''(z); z)$ are univalent functions in U and if p satisfies the second-order differential superordination

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), (z \in U), \tag{1.2}$$

then p is called a solution of the differential superordination (1.2). (If f is subordinate to g then g is superordinate to f). An analytic function q is called subordinant, of the differential superordination if $q \prec p$ for all the functions p satisfying (1.2).

An univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all the subordinants q of (1.2) is called the best subordinant. Miller and Mocanu [9] have obtained sufficient conditions on the functions h,q and ψ for which the following implication holds :

$$h(z) \prec \psi (p (z), zp'(z), z^2p''(z); z) \to q(z) \prec p(z). \qquad (1.3)$$

For $f \in A$ Al-Shaqsi [2] defined the following integral operator $T_\alpha f(z)$ defined by $T_\alpha f(z) = z + \sum_{n=2}^{\infty} \frac{(\alpha)n-1}{(c)n-1} a_n z^n$. (1.4)

Moreover, from (1.4), it follows that

$$z (T_{\alpha+1} f(z))' = c\alpha\, T_{\alpha+1} f(z) - (c\alpha - 1)\, T_\alpha f(z). \qquad (1.5)$$

Ali et al.[1] obtained sufficient conditions for certain normalized analytic functions to satisfy : $q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$, where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. Also, Tuneski [13] obtained sufficient conditions for

starlikeness of f in terms of the quantity $\frac{f''(z)\, f(z)}{(f'(z))^2}$ .Recently, Shanmugam et al. [11,12], Goyal et al .[8] , Atshan and Abbas [3] , Atshan and Jawad [5] , Atshan and Kazim [6] , and Atshan and Badawi [4] also obtained sandwich results for certain classes of analytic functions, for different conditions.

The main object here to find sufficient conditions for certain normalized analytic functions f to satisfy:

$$q_1(z) \prec \left(\frac{T_{\alpha+1} f(z)}{z}\right)^{\delta} \prec q_2(z)$$

and

$$q_1(z) \prec \left(\frac{pT_{\alpha+1} f(z) + (1-p)\, T_\alpha f(z) \prec q_2(z)}{z}\right)^{\delta} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$.

2. Preliminaries:

In order to prove our subordinations and superordinations results, we need the following definitions and lemmas.

Definition 2.1 [9] : Let Q the set of all functions $f(z)$ that are analytic and injective on $\overline{U} / E(f)$, where

$\overline{U} = U \cup \{z \in \partial U\}$, and $E(f) = \{ \mathcal{E} \in \partial U : \lim_{z\to\mathcal{E}} f(z) = \infty \}$,

and are such that $f'(z) \neq 0$ for $\varepsilon \in \partial U/E(f)$. Further, let the subclass of Q for which $f(0) = a$ be denoted by Q(a), and $Q(0)= Q_0$, $Q(1) = Q_1 = \{ f \in Q : f(0) = 1 \}$.

Lemma 2.1 [9] : Let q be univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing q(U) with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\, \phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. suppose that,

(i) Q (z) is starlike univalent in U ,

(ii) Re $\{\frac{zh'(z)}{Q(z)}\} > 0$ for $z \in U$.

If p is analytic in U with p(0) = q(0) , $p(U) \subseteq D$ and

$$\theta (p(z)) + zp'(z)\, \Phi(p(z)) \prec \theta (q(z)) + zp'(z)\, \Phi(q(z)), \qquad (2.1)$$

then $p \prec q$ and q is the best dominant of (2.1).

Lemma 2.2 [10] : Let q be convex univalent function in U and let $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C} / \{0\}$ and suppose that

$$\text{Re}\, \{ 1 + \frac{zq''(z)}{q'(z)} \} > \max \{0, - \text{Re}\,(\frac{\alpha}{\beta}) \}.$$

If p is analytic in U , and

$$\alpha p(z) + \beta zp'(z) \prec \alpha q(z) + \beta zq'(z), \qquad (2.2)$$

then $p \prec q$ and q is the best dominant of (2.2) .

Lemma 2.3 [10] : Let q be convex univalent in U and q(0) = 1 . Let $\beta \in \mathbb{C}$.Further assume that Re (β) > 0 . If $p \in H[q(0), 1] \cap Q$ and $p(z) + \beta zp'(z)$ is univalent in U, then

$$q(z) + \beta zq'(z) \prec p(z) + \beta zp'(z), \qquad (2.3)$$

which implies that $q \prec p$ and q is the best subordinant of (2.3) .

Lemma 2.4 [7] : Let q be convex univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing q(U). Suppose that

(i) Re $\{\frac{\theta'(q(z))}{\phi(q(z))}\} > 0$ for $z \in U$,

(ii) $Q(z) = zq'(z)\, \phi(q(z))$ is starlike univalent in $z \in U$.

If $p \in H[q(0), 1] \cap Q$ with $p(U) \subseteq D$, $\theta(p(z)) + zp'(z)\, \phi(p(z))$, is univalent in U and

$$\theta(q(z)) + zq'(z)\, \phi(q(z)) \prec \theta(p(z)) + zp'(z)\, \phi(p(z)), \qquad (2.4)$$

then $q \prec p$ and q is the best subordinant of (2.4) .

3. Subordination Results

Theorem3.1 : Let q be convex univalent function in U with q(0) = 1, $0 \neq \eta \in \mathbb{C}$, $\delta > 0$ and suppose that q satisfies :

$$\text{Re}\, \{ 1 + \frac{zq''(z)}{q'(z)} \} > \max\{0, - \text{Re}\,(\frac{\delta}{\eta}) \}. \qquad (3.1)$$

If $f \in A$ satisfies the subordination

$$1 - \eta (c\alpha - 1) \left(\frac{T_{\alpha+1}f(z)}{z}\right)^{\delta} + \eta (c\alpha - 1) \left(\frac{T_{\alpha+1} f(z)}{z}\right)^{\delta} \left(\frac{T_\alpha f(z)}{T_{\alpha+1} f(z)}\right) \prec q(z) + \frac{\eta}{\delta} z q'(z), \qquad (3.2)$$

then

$$\left(\frac{T_{\alpha+1} f(z)}{z}\right)^{\delta} \prec q(z) \qquad (3.3)$$

and q is the best dominant of (3.2).

Proof : Consider a function p(z) by

$$p(z) = \left(\frac{T_{\alpha+1} f(z)}{z}\right)^{\delta}. \quad (3.4)$$

Differentiating (3.4) with respect to z logarithmically , we get

$$\frac{z\,p'(z)}{p(z)} = \delta\left(\frac{zT_{\alpha+1} f(z)'}{T_{\alpha+1} f(z)} - 1\right). \quad (3.5)$$

Now , in view of (1.5) , we obtain

$$\frac{z\,p'(z)}{p(z)} = \delta\left(c\alpha\left(\frac{T_{\alpha+1} f(z)}{T_{\alpha+1} f(z)} - 1\right) + \left(\frac{T_{\alpha+1} f(z)}{T_{\alpha+1} f(z)} - 1\right)\right).$$

Therefore

$$\frac{zp'(z)}{\delta} = \left(\frac{T_{\alpha+1} f(z)}{z}\right)^{\delta}\left(c\alpha\left(\frac{T_{\alpha+1} f(z)}{T_{\alpha+1} f(z)} - 1\right) + \left(\frac{T_{\alpha+1} f(z)}{T_{\alpha+1} f(z)} - 1\right)\right).$$

The subordination (3.2) from the hypothesis becomes

$$p(z) + \frac{\eta}{\delta} z\,p'(z) \prec q(z) + \frac{\eta}{\delta} z\,q'(z).$$

An application of Lemma (2.2) with $\beta = \frac{\eta}{\delta}$ and $\alpha = 1$,we obtain(3.3).

Putting $q(z) = \left(\frac{1+z}{1-z}\right)$ in Theorem (3.1) ,we obtain the following corollary .

Corollary 3.1 , Let $0 \neq \eta \in \mathbb{C}$, $\delta > 0$ and

$$\text{Re}\left\{ 1 + \frac{2z}{1-z} \right\} > \max\left\{ 0, -\text{Re}\left(\frac{\delta}{\eta}\right) \right\}.$$

If $f \in A$ satisfies the subordination

$$1 - \eta\,(c\alpha - 1)\left(\frac{T_{\alpha+1} f(z)}{z}\right)^{\delta} + \eta\,(c\alpha - 1)\left(\frac{T_{\alpha+1} f(z)}{z}\right)^{\delta}\left(\frac{T_{\alpha} f(z)}{T_{\alpha+1} f(z)}\right) \prec \left(\frac{1-z^2+2\frac{\eta}{\delta}z}{(1-z)^2}\right),$$ then

$$\left(\frac{T_{\alpha+1} f(z)}{z}\right)^{\delta} \prec \left(\frac{1+z}{1-z}\right),$$

and $q(z) = \left(\frac{1+z}{1-z}\right)$ is the best dominant.

Theorem 3.2 : Let q be convex univalent in U with q(0) = 1 , $q(z) \neq 0$($z \in U$) and assume that q satisfies :

$$\text{Re}\left\{ 1 - \frac{\delta}{\eta} + \frac{zq''(z)}{q''(z)} \right\} > 0, \quad (3.6)$$

where $\eta \in \mathbb{C} / \{0\}$, $\lambda > 0$ and $z \in U$. Suppose that $-\eta\, z\, q'(z)$ is starlike univalent in U.If $f \in A$ satisfies :

$$\phi(\lambda,\delta,c,\eta;z) \prec \delta\, q(z) - \eta\, zq'(z), \quad (3.7)$$

where $\phi(\lambda,\delta,c,\eta;z) = \delta\left(\frac{pT_{\alpha+1} f(z)+(1-p)T_{\alpha} f(z)}{z}\right)^{\delta} -$

$$\eta\delta\left(\frac{pT_{\alpha+1} f(z)+(1-p)\,T_{\alpha} f(z)}{z}\right)^{\delta}\left(\frac{pT_{\alpha} f(z)+(1-p)T_{\alpha+1} f(z)}{pT_{\alpha+1} f(z)+(1-p)T_{\alpha+1} f(z)} - 1\right), \quad (3.8)$$

then $$\left(\frac{pT_{\alpha} f(z)+(1-p)T_{\alpha} f(z)}{z}\right)^{\delta} \prec q(z), \quad (3.9)$$

and q(z) is the best dominant of (3.7).

Proof : Consider a function p(z) by

$$p(z) = \left(\frac{pT_{\alpha+1} f(z)+(1-p)T_{\alpha} f(z)}{z}\right)^{\delta} \quad (3.10)$$

by setting ,

$$\theta(w) = \delta w \text{ and } \phi(w) = -\eta,\ w \neq 0$$

we see that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C} / \{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C} / \{0\}$. Also , we get

$$Q(z) = zq'(z)\,\phi(q(z)) = -\eta\, zq'(z),$$

and

$$h(z) = \theta(q(z)) + Q(z) = \delta\, q(z) - \eta\, zq'(z).$$

It is clear that Q(z) is starlike univalent in U.

$$\text{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \text{Re}\left\{ 1 - \frac{\delta}{\eta} + \frac{zq''(z)}{q'(z)} \right\} > 0.$$

By a straightforward computation,we obtain

$$\delta p(z) - \eta\, zp'(z) = \phi(\lambda,\delta,c,\Psi;z), \quad (3.11)$$

where $\phi(\lambda,\delta,c,\Psi;z)$ is given by (3.8) from (3.7) and (3.11) , we have $\delta p(z) - \eta\, zp'(z) \prec \delta q(z) - \eta\, zq'(z)$. (3.12)

Therefore. By Lemma (2.1) , we get $p(z) \prec q(z)$. By using (3.10), we obtain the result .

Putting $q(z) = \frac{1+Az}{1+Bz}$ ($-1 \leq B < A < 1$) in Theorem (3.2), we obtain The following Corollary.

Corollary 3.2 : Let $-1 \leq B < A \leq 1$ and

$$\text{Re}\left\{ 1 - \frac{\delta}{\eta} + \frac{z2B}{1+Bz} \right\} > 0,$$

where $\eta \in \mathbb{C}/\{0\}$ and $z \in U$, if $f \in A$ satisfies

$\phi(\lambda,\delta,c,\eta,z) \prec \left(\delta\left(\frac{1+Az}{1+Bz}\right) - \eta\, z\frac{A-B}{1+Bz)^2}\right)$ and $\phi(\lambda,\delta,c,\Psi;z)$ is given by (3.8).

$$\left(\frac{pT_{\alpha+1} f(z)+(1-p)\,T_{\alpha} f(z)}{z}\right) \prec \frac{1+Az}{1+Bz}$$

and $q(z) = \frac{1+Az}{1+Bz}$ is the best dominant.

4. Superordination Results

Theorem 4.1: Let q be convex univalent in U with q(0) =1, $\delta > 0$ and Re $\{\eta\} > 0$. Let $f \in A$ satisfies $\left(\frac{T_{\alpha+1} f(z)}{z}\right)^{\delta} \in H\,[q(0), 1] \cap Q$ and $(1-\eta\,(c\alpha))\left(\frac{T_{\alpha+1} f(z)}{z}\right)^{\delta} + \eta\,(c\alpha - 1)\left(\frac{T_{\alpha+1} f(z)}{z}\right)^{\delta}\left(\frac{T_{\alpha} f(z)}{T_{\alpha+1} f(z)}\right)$ be univalent in U . If $q(z) + \frac{\eta}{\delta} zq'(z) \prec$

$$(1-\eta\,(c\alpha))\left(\frac{T\alpha+1\, f(z)}{z}\right)^{\delta} + \eta(c\alpha-1)\left(\frac{T\alpha+1\, f(z)}{z}\right)^{\delta}\left(\frac{T\alpha\, f(z)}{T\alpha+1\, f(z)}\right), \quad (4.1)$$

then $$q(z) \prec \left(\frac{T_{\alpha+1} f(z)}{z}\right)^{\delta} \quad (4.2)$$

and q is the best subordinant of (4.1).

Proof : Consider a function p(z) by $P(z) = (\frac{T_{\alpha+1} f(z)}{z})^{\delta}$. (4.3)

Differentiating (4.3) with respect to z logarithmically , we get

$$\frac{zp'(z)}{p(z)} = \delta\,(\frac{z(T_{\alpha+1} f(z))'}{T_{\alpha+1} f(z)} - 1). \qquad (4.4)$$

After some computations and using (1.5) , from (4.4), we obtain

$$(1-\eta\,(c\alpha))\,(\frac{T_{\alpha+1} f(z)}{z})^{\delta} + \eta\,(c\alpha - 1)\,(\frac{T_{\alpha+1} f(z)}{z})^{\delta}\,(\frac{T_{\alpha} f(z)}{T_{\alpha+1} f(z)}) = p(z) + \frac{\eta}{\delta} zp'(z)$$

and now , by using Lemma (2.3) , we get the desired result.

Putting $q(z) = \frac{1+z}{1-z}$ in Theorem (4.1) ,we obtain the following Corollary.

Corollary 4.1: Let $\delta > 0$ and Re{ η } > 0 . If $f \in A$ satisfies:

$$(\frac{T_{\alpha+1} f(z)}{z})^{\delta} \in H\,[q(0)\,,1] \cap Q$$

and

$$(1-\eta\,(c\alpha))\,(\frac{T_{\alpha+1} f(z)}{z})^{\delta} + \eta\,(c\alpha - 1)\,(\frac{T_{\alpha+1} f(z)}{z})^{\delta}\,(\frac{T_{\alpha} f(z)}{T_{\alpha+1} f(z)}),$$

be univalent in U . If

$$(\frac{1-z^2+2\frac{\eta}{\delta}z}{(1-z)^2}) \prec$$

$$(1-\eta\,(c\alpha))\,(\frac{T_{\alpha+1} f(z)}{z})^{\delta} + \eta\,(c\alpha - 1)\,(\frac{T_{\alpha+1} f(z)}{z})^{\delta}\,(\frac{T_{\alpha} f(z)}{T_{\alpha+1} f(z)}),$$

then

$$(\frac{1+z}{1-z}) \prec (\frac{T_{\alpha+1} f(z)}{z})^{\delta}$$

and $q(z) = \frac{1+z}{1-z}$ is the best subordinant .

Theorem 4.2: Let q be convex univalent in U with q(0) = 1 ,and assume that q satisfies

$$\text{Re}\,\{\frac{-\delta\, q'(z)}{\eta}\} > 0, \qquad (4.5)$$

where $\eta \in \mathbb{C} / \{0\}$ and $z \in U$.

Suppose that $-\eta\, zq'(z)$ is starlike univalent in U , let $f \in A$ satisfies :

$$(\frac{pT_{\alpha+1} f(z)+(1-p)\,T_{\alpha} f(z)}{z}) \in H\,[q(0),1] \cap Q$$

and $\phi\,(\lambda,\delta,c,\eta;z)$ is univalent in U, where $\phi\,(\lambda,\delta,c,\eta;z)$ is given by (3.8) . If $\delta q(z) - \eta\, zq'(z) \prec \phi(\lambda,\delta,c,\eta;z)$, (4.6)

$$\text{then}\quad q(z) \prec (\frac{pT_{\alpha+1} f(z)+(1-p)\,T_{\alpha} f(z)}{z})^{\delta} \qquad (4.7)$$

and q is the best subordinant of (4.6).

Proof : Consider a function p(z) by

$$p(z) = (\frac{pT_{\alpha+1} f(z)+(1-p)\,T_{\alpha} f(z)}{z})^{\delta}. \qquad (4.8)$$

By setting

$$\theta\,(w) = \delta w \text{ and } \phi(w) = -\eta\,, w \neq 0,$$

we see that $\theta\,(w)$ is analytic in $\mathbb{C}$, $\phi\,(w)$ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi\,(w) \neq 0$, $w \in \mathbb{C}\,\{0\}$. Also , we get

$$Q(z) = zq'(z)\,\phi\, q(z) = -\eta\, zq'(z).$$

It is clear that Q(z) is starlike univalent in U ,

$$\text{Re}\,\{\frac{\theta'(q(z))}{\phi\,(q(z))}\} = \text{Re}\,\{\frac{-\delta\, q'(z)}{\eta}\} > 0.$$

By a straightforward computation , we obtain

$$\phi\,(\lambda,\delta,c,\eta;z) = \delta p(z) - \eta\, zp'(z), \qquad (4.9)$$

where $\phi(\lambda,\delta,c,\eta;z)$ is given by (3.8) From (4.6) and (4.9),we have

$$\delta q(z) - \eta\, zq'(z) \prec \delta p(z) - \eta\, zp'(z).$$

Therefore , by Lemma (2.4) , we get $q(z) \prec p(z)$. by using (4.8), we obtain the result.

Concluding the results of differential subordination and superordination we arrive at the following " sandwich results " .

5. Sandwich Results

Theorem 5.1 : Let q_1 be convex univalent in U with $q_1(0) = 1$, Re{ η }> 0 and let q_2 be univalent in U, $q_2(0) = 1$ and satisfies (3.1) , let $f \in A$ satisfies :

$$(\frac{T_{\alpha+1} f(z)}{z})^{\delta} \in H\,[1,1] \cap Q$$

and

$$(1-\eta\,(c\alpha))\,(\frac{T_{\alpha+1} f(z)}{z})^{\delta} + \eta\,(c\alpha - 1)\,(\frac{T_{\alpha+1} f(z)}{z})^{\delta}\,(\frac{T_{\alpha}f(z)}{T_{\alpha+1} f(z)})$$

be univalent U . If $q_1(z) + \frac{\eta}{\delta} zq'_1(z) \prec$

$$(1-\eta\,(c\alpha))\,(\frac{T_{\alpha+1} f(z)}{z})^{\delta} + \eta\,(c\alpha - 1)\,(\frac{T_{\alpha+1} f(z)}{z})^{\delta}\,(\frac{T_{\alpha} f(z)}{T_{\alpha+1} f(z)}) \prec q_2(z) + \frac{\eta}{\delta} zq'_2(z),$$

then $$q_1(z) \prec (\frac{T_{\alpha+1} f(z)}{z})^{\delta} \prec q_2(z),$$

and q_1 and q_2 are respectively , the best subordinant and the best dominant .

Theorem 5.2: Let q_1 be convex univalent in U with $q_1(0) = 1$, and satisfies (4.5). Let q_2 be univalent in U $q_2(0) = 1$, satisfies (3.6), let $f \in A$ satisfies

$$(\frac{pT_{\alpha+1} f(z)+(1-p)\,T_{\alpha}f(z)}{z})^{\delta} \in H\,[1,1] \cap Q$$

and $\phi(\lambda,\delta,c,\eta;z)$ is univalent in U , where $\phi(\lambda,\delta,c,\eta;z)$ is given by (3.8). If

$$\delta q_1(z) - \eta z q'_1(z) \prec \phi(\lambda,\delta,c,\eta;z) \prec \delta q_2(z) - \eta z q'_2(z),$$

then

$$q_1(z) \prec \left(\frac{pT_{\alpha+1} f(z) + (1-p)\, T_\alpha f(z)}{z} \right)^\delta \prec q_2(z) ,$$

and q_1 and q_2 are respectively the best subordinant and the best dominant .

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