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Differential Sandwich Results For Univalent Functions

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Abstract:
In the present paper, we obtain some subordination and superordination Results involving the integral operator \(T_\alpha\) for certain normalized analytic functions in the open unit disk. These results are applied to obtain sandwich results.

Keywords: Analytic function, differential subordination, superordination, sandwich theorem, dominant, subordinant, integral operator.

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1. Introduction:
Let \(H = H(U)\) be the class of analytic functions in the open unit disk \(U = \{z \in \mathbb{C} : |z| < 1\}\). For a positive integer and a \(a \in \mathbb{C}\), let \(H[a,n]\) be the subclass of the function \(f \in H\) consisting of functions of the form:

\[
f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots \quad (a \in \mathbb{C}, n \in \mathbb{N}).
\]

Also, let \(A\) be the subclass of \(H\) consisting of functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n.
\]

Let \(f, g \in H\). The function \(f\) is said to be subordinate to \(g\), or \(g\) is said to be superordinate to \(f\), if there exists a Schwarz function \(w\) analytic in \(U\) with \(w(0) = 0\) and \(|w(z)| < 1\) (\(z \in U\)), such that \(f(z) = g(w(z))\), in such a case, we write \(f < g\) or \(f(z) < g(z)\) (\(z \in U\)). If \(g\) is univalent in \(U\), then \(f < g\) if and only if \(f(0) = g(0)\) and \(f(U) \subset g(U)\).

Let \(p, h \in H\) and \(\psi(r, \delta, t; z) : \mathbb{C} \times \mathbb{C} \to \mathbb{C}\). If \(p\) and \(\psi(p(z), zp'(z), z^2p''(z); z)\) are univalent functions in \(U\) and if \(p\) satisfies the second-order differential superordination

\[
b(z) < \psi(p(z), zp'(z), z^2p''(z); z), (z \in U),
\]

where \(b(z) < \psi(p(z), zp'(z), z^2p''(z); z)\).
then $p$ is called a solution of the differential superordination (1.2).

If $f$ is subordinate to $g$ then $g$ is superordinate to $f$. An analytic function $q$ is called subordinant, of the differential superordination if $q < p$ for all the functions $p$ satisfying (1.2).

An univalent subordinant $q$ that satisfies $q < q$ for all the subordinates $q$ of (1.2) is called the best subordinator. Miller and Mocanu [9] have obtained sufficient conditions on the functions $h, q$ and $\psi$ for which the following implication holds:

$$h(z) < \psi (p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) < p(z).$$  \hspace{1cm} (1.3)

For $f \in A$ Al-Shaqsi [2] defined the following integral operator $T_{\alpha} f(z)$ defined by $T_{\alpha} f(z) = z + \sum_{n=2}^{\infty} \left( \frac{\alpha n}{n-1} \right)_{c} a_n z^n$. Moreover, from (1.4), it follows that

$$z (T_{\alpha+1} f(z))^\alpha = c \alpha T_{\alpha+1} f(z) - (c\alpha - 1) T_{\alpha} f(z).$$  \hspace{1cm} (1.5)

Ali et al. [1] obtained sufficient conditions for certain normalized analytic functions to satisfy:

$$q_1(z) < \frac{zf(z)}{f(z)} < q_2(z),$$

where $q_1$ and $q_2$ are given univalent functions in $U$ with $q_1(0) = q_2(0) = 1$. Also, Tuneski [13] obtained sufficient conditions for starlikeness of $f$ in terms of the quantity $I_{\alpha}^{\infty} f(z)/(F(z))$. Recently, Shannumag et al. [11, 12], Goyal et al. [8], Atshan and Abbas [3], Atshan and Jawad [5], Atshan and Kazim [6], and Atshan and Badawi [4] also obtained sandwich results for certain classes of analytic functions.

The main object here to find sufficient conditions for certain normalized analytic functions $f$ to satisfy:

$$q_1(z) < \left( \frac{T_{\alpha+1} f(z)}{z} \right)^{\delta} < q_2(z)$$

and

$$q_1(z) < \left( \frac{T_{\alpha+1} f(z) + (\alpha - 1) T_{\alpha} f(z)}{z} \right)^{\delta} < q_2(z),$$

where $q_1$ and $q_2$ are given univalent functions in $U$ with $q_1(0) = q_2(0) = 1$.

2. Preliminaries: In order to prove our subordinations and superordinations results, we need the following definitions and lemmas.

Definition 2.1 [9]: Let $Q$ the set of all functions $f(z)$ that are analytic and injective on $U$ \ensuremath{\text{and}} $\mathfrak{f}(f)$, where $U = U \cup \{z \in \partial U\}$, and $\mathfrak{f}(f) = \{E \in \partial V : \lim_{z \to \partial E} f(z) = \infty\}$. and such are that $f(z) \neq 0$ for $E \in \partial U$. Further, let the subclass of $Q$ for which $f(0) = a$ be denoted by $Q(a)$, and $Q(0) = Q_0, Q(1) = Q_1 = \{f \in Q : f(0) = 1\}$.

Lemma 2.1 [9]: Let $q$ be univalent in the unit disk $U$ and let $\theta$ and $\phi$ be analytic in a domain $D$ containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = q^*(z) \phi (q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that,

(i) $Q(z)$ is starlike univalent in $U$.
(ii) $Re \left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ for $z \in U$.

If $p$ is analytic in $U$ with $p(0) = q(0), p(U) \subset D$ and

$$\theta(p(z)) + zp'(z) \Phi(p(z)) < \theta(q(z)) + zp'(z) \Phi(q(z)).$$  \hspace{1cm} (2.1)

then $p < q$ and $q$ is the best dominant of (2.1).

Lemma 2.2 [10]: Let $q$ be convex univalent function in $U$ and let $\alpha \in \mathbb{C}, \beta \in \mathbb{C}$ with $0 < \beta < 1$. If $p \in H[q(0), 1] \cap Q$ and $p(z) + \beta zp'(z)$ is univalent in $U$, then

$$q(z) + \beta zq'(z) < p(z) + \beta zp'(z),$$  \hspace{1cm} (2.3)

which implies that $q < p$ and $q$ is the best subordinant of (2.3).

Lemma 2.3 [10]: Let $q$ be convex univalent in $U$ and $q(0) = 1$. Let $\beta \in \mathbb{C}$. Further assume that $Re \{\beta\} > 0$. If $p \in H[q(0), 1] \cap Q$ and $p(z) + \beta zp'(z)$ is univalent in $U$, then

$$\theta(q(z)) + zq'(z) \Phi(q(z)) < \theta(p(z)) + zp'(z) \Phi(p(z)).$$  \hspace{1cm} (2.4)

then $q < p$ and $q$ is the best subordinant of (2.4).

3. Subordination Results

Theorem 3.1: Let $q$ be convex univalent function in $U$ with $q(0) = 1, 0 \neq \eta \in \mathbb{C}, \delta > 0$ and suppose that $q$ satisfies:

$$Re \left\{1 + \frac{zq'(z)}{q(z)}\right\} > \max \{0, -\Re \left(\frac{\alpha}{\beta}\right)\}.$$  \hspace{1cm} (3.1)

If $f \in A$ satisfies the subordination

$$1 - \eta (\alpha - 1) \left( \frac{T_{\alpha+1} f(z)}{z} \right)^{\delta} + \eta (\alpha - 1) \left( \frac{T_{\alpha+1} f(z)}{z} \right) \phi \left( \frac{T_{\alpha+1} f(z)}{T_{\alpha+1} f(z)} \right) < q(z) + \frac{\beta z}{\alpha} q'(z),$$  \hspace{1cm} (3.2)

then

$$\left( \frac{T_{\alpha+1} f(z)}{z} \right)^{\delta} < q(z),$$  \hspace{1cm} (3.3)

and $q$ is the best dominant of (3.2).
\textbf{Proof :} Consider a function \( p(z) \) by
\[
p(z) = \left( \frac{T_{\alpha+1}(f(z))}{z} \right)^{\delta}.
\] (3.4)
Differentiating (3.4) with respect to \( z \) logarithmically, we get
\[
\frac{z p'(z)}{p(z)} = \delta \left( \frac{T_{\alpha+1}(f(z))}{T_{\alpha+1}(f(z))} \right) - 1.
\] (3.5)
Now, in view of (1.5), we obtain
\[
\frac{z p'(z)}{p(z)} = \delta \left( \frac{T_{\alpha+1}(f(z))}{T_{\alpha+1}(f(z))} + 1 \right) + \left( \frac{T_{\alpha+1}(f(z))}{T_{\alpha+1}(f(z))} - 1 \right).
\]
Therefore
\[
\frac{z p'(z)}{p(z)} = \left( \frac{T_{\alpha+1}(f(z))}{z} \right)^{\delta} \left( \frac{T_{\alpha+1}(f(z))}{T_{\alpha+1}(f(z))} + 1 \right) + \left( \frac{T_{\alpha+1}(f(z))}{T_{\alpha+1}(f(z))} - 1 \right).
\]
The subordination (3.2) from the hypothesis becomes
\[
p(z) + \frac{\eta}{\delta} p(z) < q(z) + \frac{\eta}{\delta} z q'(z).
\]
An application of Lemma (2.2) with \( \beta = \frac{\eta}{\delta} \) and \( \alpha = 1 \), we obtain (3.3).

\textbf{Corollary 3.1 :} Let \( 0 \neq \eta \in \mathbb{C}, \delta > 0 \) and
\[
\text{Re} \left( 1 + \frac{\eta z}{1-z} \right) > \max \{ 0, - \text{Re} \left( \frac{\eta}{\delta} \right) \}.
\]
If \( f \in A \) satisfies the subordination
\[
1 - \eta \left( 1, 1 \right) \left( \frac{T_{\alpha+1}(f(z))}{z} \right)^{\delta + \eta} \left( \frac{T_{\alpha+1}(f(z))}{z} \right)^{\delta} \left( \frac{T_{\alpha+1}(f(z))}{z} \right) < \frac{1-z^{2}+2z}{1-z^{2}},
\]
then
\[
\left( \frac{T_{\alpha+1}(f(z))}{z} \right)^{\delta} < \frac{1+z}{1-z}.
\]
and \( q(z) = \left( \frac{1+z}{1-z} \right) \) is the best dominant.

\textbf{Theorem 3.2 :} Let \( q \) be convex univalent in \( U \) with \( q(0) = 1 \), \( q(z) \neq 0 \) in \( U \) and assume that \( q \) satisfies:
\[
\text{Re} \left( 1 + \frac{\eta z}{1-z} \right) > 0,
\] (3.6)
where \( \eta \in \mathbb{C}/\{0\}, \lambda > 0 \) and \( z \in U \). Suppose that \( \eta z \) is starlike univalent in \( U \). If \( f \in A \) satisfies:
\[
\text{Re} \left( \frac{p_{\alpha+1}(f(z))}{z} \right)^{\delta} = \eta \left( \frac{p_{\alpha+1}(f(z))}{z} \right)^{\delta} + \eta \left( \frac{p_{\alpha+1}(f(z))}{z} \right)^{\delta} \left( \frac{p_{\alpha+1}(f(z))}{z} \right) - 1,
\] (3.8)
then
\[
\left( \frac{p_{\alpha+1}(f(z))}{z} \right)^{\delta} < \left( \frac{T_{\alpha+1}(f(z))}{z} \right)^{\delta} \]
and \( q(z) \) is the best dominant of (3.7).

\textbf{Proof :} Consider a function \( p(z) \) by
\[
p(z) = \left( \frac{p_{\alpha+1}(f(z))}{z} \right)^{\delta}.
\] (3.10)
by setting,
\[
\theta(z) = \delta w + \phi(w) = -\eta, w \neq 0
\]
we see that \( \theta(w) \) is analytic in \( C, \phi(w) \) is analytic in \( C/\{0\} \) and that \( \phi(w) \neq 0 \), \( w \in C/\{0\} \). Also, we get
\[
Q(z) = \frac{q(z)}{\phi(q(z))} = -\eta z q'(z),
\]
and
\[
h(z) = \theta(q(z)) + Q(z) = \delta q(z) - \eta z q'(z).
\]
It is clear that \( Q(z) \) is starlike univalent in \( U \).
\[
\text{Re} \left( \frac{z h(z)}{Q(z)} \right) = \text{Re} \left( \frac{1}{\eta} + \frac{z q'(z)}{q(z)} \right) > 0.
\]
By a straightforward computation, we obtain
\[
\delta p(z) - \eta z p'(z) = \phi(\lambda, \delta, c, \Psi, z),
\] (3.11)
where \( \phi(\lambda, \delta, c, \Psi, z) \) is given by (3.8) from (3.7) and (3.11), we have
\[
\delta p(z) - \eta z p'(z) < \delta q(z) - \eta z q'(z).
\] (3.12)
Therefore, By Lemma (2.1), we get \( p(z) < q(z) \). By using (3.10), we obtain the result.

Putting \( q(z) = \left( \frac{1+z}{1-z} \right) \) in Theorem (3.2), we obtain The following Corollary.

\textbf{Corollary 3.2 :} Let \( -1 \leq B < A < 1 \) and
\[
\text{Re} \left( \frac{1}{\eta} + \frac{z q'(z)}{q(z)} \right) > 0,
\]
where \( \eta \in \mathbb{C}/\{0\} \) and \( z \in U \). If \( f \in A \) satisfies
\[
\phi(\lambda, \delta, c, \eta, z) < \delta \left( \frac{1+z}{1-z} \right) - \eta z \frac{A-B}{1+Bz} \) \) and \( \phi(\lambda, \delta, c, \Psi, z) \) is given by (3.8).
\[
\left( \frac{p_{\alpha+1}(f(z))+(1-p)T_{\alpha+1}(f(z))}{z} \right) < \frac{1+z}{1-z},
\]
and \( q(z) = \left( \frac{1+z}{1-z} \right) \) is the best dominant.

\section{Superordination Results}

\textbf{Theorem 4.1 :} Let \( q \) be convex univalent in \( U \) with \( q(0) = 1, \delta > 0 \) and \( \text{Re} \left( \frac{p_{\alpha+1}(f(z))}{z} \right) > 0 \) and \( \text{Re} \left( \frac{1}{\eta} \right) > 0 \). Let \( f \in A \) satisfies \( \left( \frac{p_{\alpha+1}(f(z))}{z} \right)^{\delta} \in H \), \( q(0) \), \( 1 \) \( \cap \) \( Q \) and
\[
(1-\eta)(1-\eta) \left( \frac{T_{\alpha+1}(f(z))}{z} \right)^{\delta} + \eta \left( \frac{T_{\alpha+1}(f(z))}{z} \right)^{\delta} \left( \frac{T_{\alpha+1}(f(z))}{z} \right)^{\delta} \left( \frac{T_{\alpha+1}(f(z))}{z} \right) \}
\]
be univalent in \( U \). If \( q(z) + \frac{\eta z}{1-z} < \left( \frac{p_{\alpha+1}(f(z))}{z} \right)^{\delta} \left( \frac{T_{\alpha+1}(f(z))}{z} \right)^{\delta} \left( \frac{T_{\alpha+1}(f(z))}{z} \right) \}
\]
then
\[
q(z) < \left( \frac{T_{\alpha+1}(f(z))}{z} \right)^{\delta} \left( \frac{T_{\alpha+1}(f(z))}{z} \right)^{\delta} \left( \frac{T_{\alpha+1}(f(z))}{z} \right) \}
\] (4.1)
then
\[
q(z) < \left( \frac{T_{\alpha+1}(f(z))}{z} \right)^{\delta} \left( \frac{T_{\alpha+1}(f(z))}{z} \right)^{\delta} \left( \frac{T_{\alpha+1}(f(z))}{z} \right) \}
\] (4.2)
and \( q \) is the best subordinant of (4.1).

**Proof:** Consider a function \( p(z) \) by \( P(z) = \left( \frac{T_{n+1} f(z)}{z} \right)^\delta \). (4.3)

Differentiating (4.3) with respect to \( z \) logarithmically, we get
\[
\frac{z p'(z)}{p(z)} = \delta \left( \frac{T_{n+1} f(z)}{T_{n+1} f(z)} \right)' - 1.
\] (4.4)

After some computations and using (1.5), from (4.4), we obtain
\[
(1 - \eta (\alpha)) \left( \frac{T_{n+1} f(z)}{z} \right)^\delta + \eta (\alpha - 1) \left( \frac{T_{n+1} f(z)}{z} \right)^\delta \left( \frac{T_{n+1} f(z)}{T_{n+1} f(z)} \right) = P(z) + \frac{\eta}{\delta} z p'(z)
\]
and now, by using Lemma (2.3), we get the desired result.

Putting \( q(z) = \frac{1 + z}{1 - z} \) in Theorem (4.1), we obtain the following Corollary.

**Corollary 4.1:** Let \( \delta > 0 \) and \( \Re \{ \eta \} > 0 \). If \( f \in A \) satisfies:
\[
\left( \frac{T_{n+1} f(z)}{z} \right)^\delta \in H[q(0), 1] \cap Q
\]
and
\[
(1 - \eta (\alpha)) \left( \frac{T_{n+1} f(z)}{z} \right)^\delta + \eta (\alpha - 1) \left( \frac{T_{n+1} f(z)}{z} \right)^\delta \left( \frac{T_{n+1} f(z)}{T_{n+1} f(z)} \right),
\]
be univalent in \( U \). If
\[
1 - z^2 + 2 z \frac{\delta}{(1 - z)^2} < (1 - \eta (\alpha)) \left( \frac{T_{n+1} f(z)}{z} \right)^\delta + \eta (\alpha - 1) \left( \frac{T_{n+1} f(z)}{z} \right)^\delta \left( \frac{T_{n+1} f(z)}{T_{n+1} f(z)} \right),
\]
then
\[
\left( \frac{1 + z}{1 - z} \right) < \left( \frac{T_{n+1} f(z)}{z} \right)^\delta
\]
and \( q(z) = \frac{1 + z}{1 - z} \) is the best subordinant.

**Theorem 4.2:** Let \( q \) be convex univalent in \( U \) with \( q(0) = 1 \), and assume that \( q \) satisfies:
\[
\Re \{ -\delta q'(z) \eta \} > 0,
\] (4.5)
where \( \eta \in \mathbb{C} \setminus \{ 0 \} \) and \( z \in U \).

Suppose that \( -\eta q'(z) \) is starlike univalent in \( U \), let \( f \in A \) satisfies:
\[
\left( \frac{p T_{n+1} f(z) + (1 - P) T_{n+1} f(z)}{z} \right) \in H[q(0), 1] \cap Q
\]
and \( \phi(\lambda, \delta, c, \eta; z) \) is univalent in \( U \), where \( \phi(\lambda, \delta, c, \eta; z) \) is given by (3.8). If \( \delta q(z) - \eta z q'(z) < \phi(\lambda, \delta, c, \eta; z) \),
\[
(1 - \eta (\alpha)) \left( \frac{T_{n+1} f(z)}{z} \right)^\delta + \eta (\alpha - 1) \left( \frac{T_{n+1} f(z)}{z} \right)^\delta \left( \frac{T_{n+1} f(z)}{T_{n+1} f(z)} \right) < q(z) + \frac{\eta}{\delta} z q'(z) \)
then
\[
q_1(z) < \left( \frac{T_{n+1} f(z)}{z} \right)^\delta < q_2(z),
\]
and \( q_1 \) and \( q_2 \) are respectively, the best subordinant and the best dominant.

**Theorem 5.2:** Let \( q_1 \) be convex univalent in \( U \) with \( q_1(0) = 1 \), and satisfies (4.5). Let \( q_2 \) be univalent in \( U \), \( q_2(0) = 1 \), satisfies (3.6), let \( f \in A \) satisfies:
\[
\left( \frac{p T_{n+1} f(z) + (1 - P) T_{n+1} f(z)}{z} \right) \in H[1,1] \cap Q
\]

**Proof:** Consider a function \( p(z) \) by
\[
p(z) = \left( \frac{p T_{n+1} f(z) + (1 - P) T_{n+1} f(z)}{z} \right)^\delta.
\] (4.8)

By setting \( \theta(z) = \delta w \) and \( \phi(w) = -\eta w \neq 0 \), we see that \( \theta(z) \) is analytic in \( C, \phi(w) \) is analytic in \( C \setminus \{ 0 \} \) and that \( \phi(w) \neq 0, w \in C \setminus \{ 0 \} \). Also, we get
\[
Q(z) = z q'(z) \phi(z) q(z) = -\eta z q'(z).
\]
It is clear that \( Q(z) \) is starlike univalent in \( U \),
\[
\Re \left\{ \frac{\delta q(z)}{\phi(z)} \right\} = \Re \left\{ -\frac{\delta q'(z)}{\eta} \right\} > 0.
\]

By a straightforward computation, we obtain
\[
\phi(\lambda, \delta, c; z) = \delta p(z) - \eta z p'(z),
\] (4.9)
where \( \phi(\lambda, \delta, c; z) \) is given by (3.8). From (4.6) and (4.9), we have
\[
\delta q(z) - \eta z q'(z) < \delta p(z) - \eta z p'(z).
\]
Therefore, by Lemma (2.4), we get \( q(z) < p(z) \). By using (4.8), we obtain the result.

Concluding the results of differential subordination and superordination we arrive at the following "sandwich results".

5. Sandwich Results

**Theorem 5.1:** Let \( q_1 \) be convex univalent in \( U \) with \( q_1(0) = 1 \), \( \Re \{ \eta \} > 0 \) and let \( q \) be univalent in \( U \), \( q_2(0) = 1 \) and satisfies (3.1), let \( f \in A \) satisfies:
\[
\left( \frac{T_{n+1} f(z)}{z} \right)^\delta \in H[1,1] \cap Q
\]
and
\[
(1 - \eta (\alpha)) \left( \frac{T_{n+1} f(z)}{z} \right)^\delta + \eta (\alpha - 1) \left( \frac{T_{n+1} f(z)}{z} \right)^\delta \left( \frac{T_{n+1} f(z)}{T_{n+1} f(z)} \right) < q_1(z) + \frac{\eta}{\delta} z q_1'(z) \)
then
\[
q_1(z) < \left( \frac{T_{n+1} f(z)}{z} \right)^\delta < q_2(z),
\]
and \( q_1 \) and \( q_2 \) are respectively, the best subordinant and the best dominant.

**Theorem 5.2:** Let \( q_1 \) be convex univalent in \( U \) with \( q_1(0) = 1 \), and satisfies (4.5). Let \( q_2 \) be univalent in \( U \), \( q_2(0) = 1 \), satisfies (3.6), let \( f \in A \) satisfies:
\[
\left( \frac{p T_{n+1} f(z) + (1 - P) T_{n+1} f(z)}{z} \right) \in H[1,1] \cap Q
\]
and \( \phi (\lambda, \delta, c; \eta; z) \) is univalent in \( U \), where \( \phi (\lambda, \delta, c; \eta; z) \) is given by (3.8). If
\[
\delta q_1(z) - \eta q_1'(z) < \phi (\lambda, \delta, c; \eta; z) < \delta q_2(z) - \eta q_2'(z).
\]
then
\[
q_1(z) < \left( \frac{p T_{a_1}(f) + (1-p) T_{a_2}(f)}{z} \right)^\delta < q_2(z),
\]
and \( q_1 \) and \( q_2 \) are respectively the best subordinant and the best dominant.

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