Interpolation in Non-Continuous Functions Spaces

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REVIEW

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Abstract

In this research I will introduce a new method to an approximation in non-continuous functions space and estimate a degree of this approximation (Note that this function must be bounded) by define a new norm depending on union it together with a continuous function (I call it the compensation function) where the resulting functions from this union must be a continuous. This is possible for any continuous function but in this papers I will choose this function as an algebraic polynomial because our know of properties of polynomials in approximation theory Specifically, I will choose it as an interpolation polynomial according to Whitney's theorem in interpolation theory which we know precisely the degree of error when approximating our function by it according to Whitney's theorem, Also the range set for it, which is, as is the case in all polynomials, every real numbers, which will ensure that this set of images is closed to the subtraction process, which we will need in our work.

Keywords: Non-continuous functions, Bounded functions, Compact sets, Whitney's theorem, Whitney's constant in interpolation

1. Introduction

It is known that in approximation theory non-continuous functions say $f(x)$ cannot be approximated by suitable polynomial because there is no norm treat problem of the points of non-continuity in its domain since if this norm is in the form of an integral then no integral for non-continuous function also if the norm as a supremum, no supremum of non-continuous functions, and so on. Like approximation of unbounded functions that was approximated by a suitable norm (which called weighted norm) that treat unboundedness of this function with multiply or divide it by appropriate function called weight function such that the resulting function be a bounded function [1,2] … etc., which addresses the issue of non-boundedness. In this papers I will try to approximate non-continuous functions by a norm that treat the problem of their non-continuity points, this includes completing the set of non-continuity points from a continuous function in the set of these points, and I will naming this function by the completion function say $g(x)$.This completion process is done under specific norm which symbol by $\|f\|_{p.g}$:

$\|f\|_{p.g} = \|(f \cup g)(x)\|_p$ Now. I will define our union as: $(f \cup g)(x) = \begin{cases} f(x) & if\ x \notin I \\ g(x) & if\ x \in I \end{cases}$ where I is a non-continuous set of the function f and I will assume that the set I is of zero measure, For the completion function g it is clear that it's range is the real numbers R This ensures that the range set is a closed on subtraction operation. It is known that the polynomials, whether algebraic or trigonometric, have many characteristics and advantages that known to us, including they are continuous and derivative functions, and their graph also. So I will choose the completion function g as an algebraic polynomial $P_n \in \boldsymbol{P_n}$ where $\boldsymbol{P_n}$ the space of algebraic polynomials of degree. Now we know many properties of the polynomials (completion function), and I will symbol it by $P_n(x)$ as like of definition of $f \cup g$.

So, I define $(f \cup P_n)(x) = \begin{cases} f(x) & if\ x \notin I \\ P(x) & if\ x \in I \end{cases}$

As I mentioned earlier, I will choose the polynomial $P(x)$ as Whitney's polynomial and according

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to the Whitney's theorem in interpolation theory we know properties of this polynomial in the approximation theory.

It is important that all of this be within the definition of the intended norm and I will use supremum norm, this mean $\|f(x)\| = \sup|f(x)|$, $x \in X$. And for L_p-norm where:

$\|f(x)\|_p = (\int_a^b |f(x)|^p dx)^{\frac{1}{p}}$ it clear $\|f(x)\|_p \le \|f(x)\|$ [3] (1.1)

Now, let's remember our information about the binary operations:

let $A.B$ be any two sets, it's known that:

$A * B = \{a * b : a \epsilon A, b \epsilon B\}$ where $*$ be any binary operation [4].

For example, if we take $A = B = R$ (where R be the real numbers) then:

$R - R = \{r_1 - r_2 = r \ where \ r_1.r_2 \in R\}$ this mean every element in the set R can be writes as subtract of two numbers (notice that R closed over subtraction operation).

Also, we must remember that union of two functions is not function in general.

2. Auxiliary results

As be mention in weighted approximation the weight function is distributed in the

Its own norm (i.e. $\|f - P_n\|_{p.\alpha} = \|(f - P_n)\omega_\alpha\|_p = \|f\omega_\alpha - P\omega_\alpha\|_p$) where the

Symbol ω_α is weight function which dependent on a constant α [1,2] … etc. But that is not easy in this kind of approximation (non-continuous approximation), So before try to approximate this kind of functions I put a condition that the rang set of the function g (or the polynomial P_n) must be the real numbers R this condition ensures that this rang is a closed set on subtraction operation this mean that if $g : X \longrightarrow Y$, $\dot{y} \in g(x).x \in X$ and if $\dot{y} = y_1 - y_2$ then $y_1 \in g(x)$ and $y_2 \in g(x), x \in X$ and this property is used in proof of theorem 2.1. If we replace g by $P_n \in \ \boldsymbol{P_n}$ the above condition be content since this is one of the properties of polynomials. In the following, I will study possibility of providing the compact set of an element to best approximation of a function which belong in it, all of this is preceded by the definition of the new norm.

Before I site this norm I will prove the following necessary theorem:

2.1. Theorem

Suppose that $f : X \longrightarrow Y$ such that f be a non-continuous functions in the set of points I (where I is of zero measure) and suppose that $g, P_n : R \longrightarrow R$, where P_n is a polynomial of degree n Define in X then:

$$[(f - P_n) \cup g](x) = [f \cup g](x) - [P_n \cup g](x),$$

Proof:

Let $y \in Y$

In the **first part** I will proof this theorem if $y \in f(x)$, $x \notin I$

Necessity

Suppose that $\in [(f - P_n) \cup g](x)$.

Dependent on the definition of union either $y \in (f - P_n)(x)$ or $y \in g(x)$

Since by hypothesis if we write $y = y_1 - y_2$

Then $y = y_1 - y_2 \in (f - P_n)(x)$ this mean $y_1 \in f$ $and \ y_2 \in P_n$.

And then, $y_1 \in f \cup g$ and $y_2 \in P_n \cup g$.

So, $y = y_1 - y_2 \in (f \cup g) - (P_n \cup g)$.

If $y \in g(x)$, it is clear.

Sufficiency

Suppose that $y = y_1 - y_2 \in (f \cup g) - (P_n \cup g)$ then $y_1 \in (f \cup g)$ and $y_2 \in (P_n \cup g)$

If $y_1 \in f$ and $y_2 \in P_n$ then $y = y_1 - y_2 \in (f - P_n)$

And so $y = y_1 - y_2 \in (f - P_n) \bigcup g$

If $y_1 \in g$ and $y_2 \in g$ then $y = y_1 - y_2 \in g$

And so $y = y_1 - y_2 \in (f - P_n) \bigcup g$.

If $y_1 \in f$ and $y_2 \in g$ then $y_2 \in (f \bigcup g)$ also.

So, $y = y_1 - y_2 \in (f \bigcup g)$, since f is not continuous in some pointes

Then by closer on subtraction $y = y_1 - y_2$ must be belong in g

So, $y \in (f - P_n) \cup g$, The only remainder possibility that $y_1 \in g$ and $y_2 \in P_n$ also

Since codomain of g is R then $y_2 \in g$ also

And since g is closed on subtraction then $y = y_1 - y_2 \in g$ and then $y \in (f - P_n) \cup g$.

In the **second part** I will proof this theorem if $y \in g(x), x \in I$ (this mean $y \notin f(x)$)

Necessity

Let $y \in [(f - P_n) \cup g](x)$, $x \in I$

then y belong only in g.

since the set g closed on subtraction operation

then we can write $y = y_1 - y_2, y \in g$.

Since y arbitrary, So every element in g can be written as subtract two elements in g.

So, $y_1 \in g$ and $y_1 \in g$.

Thus $y_1 \in f \cup g$ and $y_2 \in P_n \cup g$.

And so, $y = y_1 - y_2 \in (f \cup g) - (P_n \cup g)$

Sufficiency

Suppose that $y = y_1 - y_2 \in (f \cup g) - (P_n \cup g)$

Then $y = y_1 - y_2 \in g - g = g$

So, $y \in [(f - P_n) \cup g](x)$.

And the proof is complete

Now, as I mentioned earlier and depending on the previous theorem I will define the new norm as:

$$\|f - P_n\|_{p.g} = \|(f - P_n) \cup g\|_p = \|(f \cup g) - (P_n \cup g)\|_p$$

2.1.1

So, if we replaced the function $g(x)$ in 2.1.1 by the polynomial P_n we get;

$$\|f - P_n\|_{p.g} = \|(f - P_n) \cup P_n\|_p = \|(f \cup P_n) - (P_n \cup P_n)\|_p = \|(f \cup P_n) - P_n\|_p$$

2.1.2

2.2. Theorem [5]

Let U be a compact set in a metric space X. Then, for every f in X, there exists an element of best approximation.

It should also be remembered that the image of the compact set under the influence of the continuous function is also a compact set.

2.3. Theorem

Suppose that $f: X \longrightarrow Y$ be a non-continuous functions in a set of points I and suppose that A be a compact set and $A \subset Y$ such that $f(x) \subset A, x \in X$ then A (or a compact set) provide a best approximation element of f.

Proof

Suppose that $x_n \epsilon X$ such that $x_n \to x, n \epsilon N$

Then there are two possibilities, either $x_n \in I$ or $x_n \notin I$

First If $x_n \notin I$ then $f(x_n) \in f(X)$ and then $f(x_n) \in A$

And then by above theorem A provide a best approximation element of f.

Second If $x_n \in I$ then $f(x_n)$ not found in A (i.e. $f(x_n) \notin f(X)$)

let $P_m(x)$ be an interpolation polynomial of f in the set I.

Then there are two possibilities either $f(x_n) = P_n(x_n) \in A$ or $f(x_n) = P_n(x_n) \notin A$

If $f(x_n) = P_n(x_n) \in A$

Then by above theorem A provide a best approximation element of $P_n(x_n) x \in X$

Since $f(x) = P_n(x), x \in I$ by definition of our union

Then A provide a best approximation element of $f(x), x \in I$

If $f(x_n) = P_n(x_n) \notin A$ and since the set I is of zero measure

This mean I is countable set and then it is a compact set also (with (R, T_u) real numbers with usual topology.

Since P_n is continuous function then $P_n(x_n), x_n \in I$ is a compact set also.

So, by above theorem it is provide a best approximation element of $P_n(x)$

And then it provide a best approximation element of f.

3. Main results

In this part, and in the beginning I said, I would try to find the error between non-continuous function and polynomial of best approximation according to Whitney's theorem. After that I will try to estimate the norm of non-continuous function f in its continuity region. And norm of its polynomial of best approximation by Whitney's theorem as we mentioned earlier.

In an approximation of continuous function by polynomials that studied by several scientists one of the good results for this approximation of function like f on a closed interval by a polynomial P_n is: $\|f - P_n\|_p \leq c\omega\left(f, \frac{1}{n}\right)$ [6] And here we will get a result similar to this result in non-continuous functions space, before that I will set Whitney's theorem in interpolation.

3.1. Theorem: Whitney [3]

For each $m \geq 1$ there is a number W_n with the following property. For any interval Δ and for any continuous function f on Δ there is a polynomial P of degree at most $m - 1$ such that:

$$|f - P(x)| \leq W_n \omega_n(f, \Delta), x \in \Delta.$$

Where W_n is called Whitney's constant. There are many attempts to estimate the value of this constant by several scholars and researchers for example Sendov and Ivanov and Takev …. etc., but The best of these attempts was by Binev that improved this constant to $W_n = o(n)$ [3].

Now, for example when $\Delta = [0, 1]$ we will consider the polynomial $P = P(f)$ which interpolate f at n equally spaced points:

$P\left(\frac{v}{n-1}\right) = f\left(\frac{v}{n-1}\right), v = 0, 1, \ldots\ldots.., n - 1.$ polynomial P satisfy Whitney's theorem [3].

Of cores one can put degree of above polynomial P as $m - 1 = n$ which we used in following theorem.

Also if we think of broadening (generalizing) above equally spaced points And contained in Whitney's theorem in interval $[0, 1]$ for any interval $[a, b]$.

So, I put the following rule:

$$P\left(\frac{v}{n-1}\right)=f\left(\frac{v}{n-1}\right), v=a(n-1), a(n-1)+1, \ldots\ldots.(b-1)(n-1), b(n-1) \quad (3.1.1)$$

which represent the equally spaced points in any interval $[a, b]$.

Also, let we put $\Delta = \frac{W_n}{n}$ where the points of non-continuity of the function must be within this interval.

We recall what was previously defined, if we have non-continuous function f on the set I of zero measure, and let P be the polynomial which interpolate f at points of I.This mean suppose that:

$f(x) = P_n(x), x \in I$ As we mentioned earlier $(f \cup P_n)(x) = \begin{cases} f(x) & \text{if } x \notin I \\ P_n(x) & \text{if } x \in I \end{cases}$ And then $f\cup$ P_n is a continuous function

Now, if we applied Whitney's theorem on the continuous function $f \cup P_n$ we get the following theorem:

3.2. Theorem

If f is non-continuous in the set I and $\mathrm{P}_n \in$ $\boldsymbol{P_n}$ which satisfy (3.1.1) then:

$$\|f - P_n\|_{p.g} \le W_n \omega_n\left(f.\frac{1}{n}\right).$$

Proof:

$$\|f - P_n\|_{p.g} = \|f - P_n\|_{p.P_n} = \|(f \cup P_n) - P_n\|_p$$

Now by definition of $f \cup P_n = \begin{cases} f(x) & \text{if } x \notin I \\ P_n(x) & \text{if } x \in I \end{cases}$

Then: $\|(f \cup P_n) - P_n\|_p = \|f - P_n\|_p \quad \text{if } x \notin I$

Since f is continuous in the set $x \notin I$ and by theorem 3.1 we get:

$$|f - \mathrm{P}_n(\mathrm{x})| \le W_n \omega_n(f.\Delta),$$

If we take supremum for both sides in the contrast above inequality, then we have:

$$\underbrace{\sup}_{x \in \Delta} |f - \mathrm{P}_n(\mathrm{x})| \le W_n \underbrace{\sup}_{x \in \Delta} \omega_n(f.\Delta), x \in \Delta$$

And by use properties of supremum of sets and moduli of continuity in the right side we get: $\underbrace{\sup}_{x \in \Delta} \omega_n(f.\Delta) = \underbrace{\sup}_{x \in \Delta} \left(\underbrace{sup}_{\substack{x \in \Delta, t \\ |t| \le h}} |\Delta_t^r f(x)| \right) = \underbrace{sup}_{\substack{x \in \Delta, t \\ |t| \le h}} |\Delta_t^r f(x)| =$

$\omega_n(f.\Delta)$,

And by applied definition of moduli of continuity in lift side the last inequality became:

$$\|f - \mathrm{P}_n(\mathrm{x})\| \le W_n \omega_n(f.\Delta), x \in \Delta$$

So, by 1.1 we get:

$$\|f - P_n\|_{p.g} = \|f - P_n\|_p \le W_n \omega_n\left(f.\frac{1}{n}\right)$$

if $x \in I$ and if I is of zero measure and by definition of $f \cup P_n$.

Then $\|f - P_n\|_{p.g} = \|f - P_n\|_{p.P_n} = \|(f \cup P_n) - P_n\|_p = \|P_n - P_n\|_p = 0$

Since $\lim_{n \to \infty} \omega(\mathrm{f}.\frac{1}{\mathrm{n}}) = 0$

Then $\|f - \mathrm{P_n}\|_{\mathrm{p.g}} \le W_n \omega_n(f.\frac{1}{n})$

On the other hand for constant W_n from our information of a good estimate of Whitney's constant that is $W_n = o(n)$ So $\frac{W_n}{n} \longrightarrow 0, As\ n \longrightarrow \infty$

So this regard is a good result for an approximating of non-continuous function

In following prat, I will try to estimate norm of the polynomial $\|f - P_n\|_{p.g}$ and $\|P_n(x)\|_{\mathrm{p}}$ this mean we estimate $\|f(x)\|_{\mathrm{p}}, x \notin I$ and $\|P_n(x)\|_{\mathrm{p}}, x \in I$ to confirm that $\|P_n(x)\|_{\mathrm{p}}, x \in I$ that only increases by a very small amount on $\|f(x)\|_{\mathrm{p}}, x \notin I$ by measuring the maximum breadth of norm $\|P_n(x)\|_{\mathrm{p}}$ in all domain R to ensure the smoothness of the function resulting from our union.

3.3. Theorem [3]

For each bounded function f the following trivial estimate hold:

$$\omega_n(f, \Delta) \le 2^n \|f\|_{c_\Delta}$$

3.4. Theorem

Let f be a non-continuous function and $P_n(x)$ be a Whitney's polynomial for f then: $\|f(x), x \notin I\|_{\mathrm{p}}, \le c\|P_n(x), x \in I\|_{\mathrm{p}}$ where the set I is of measure zero.

Proof:

Since by theorem 3.1 $|f \bigcup \mathrm{P}_n(x) - \mathrm{P}_n(\mathrm{x})| \le W_n \omega_n(f, \Delta), x \in \Delta$.

By early definition of union $f \cup P_n = \begin{cases} f(x) & \text{if } x \notin I \\ P_n(x) & \text{if } x \in I \end{cases}$

So:

$|f - P_n(x)| \le W_n \omega_n(f, \Delta)$ then:

$\underbrace{\sup}_{x \in \Delta} |f - \mathrm{P}_n(\mathrm{x})| \le W_n \underbrace{\sup}_{x \in \Delta} \omega_n(f, \Delta), x \in \Delta$.and by use properties of supremum of sets we get: $\|f - P_n(x)\| \le W_n \omega_n(f.\Delta), x \in \Delta$ and by use properties of norm we get:

$$\left|\|f\| - \|P_n(\mathrm{x})\|\right| \leq \|f - \mathrm{P}(\mathrm{x})\| \leq W_n\omega_n(f.\Delta), x \in \Delta$$

and then:

$$\|f\| - \|P_n(\mathrm{x})\| \leq \left|\|f\| - \|P_n(\mathrm{x})\|\right| \leq \|f - P_n(\mathrm{x})\| \leq W_n\omega_n(f.\Delta), x \in \Delta$$

So, $\|f\| - \|P_n(x)\| \leq W_n\omega_n(f.\Delta), x \in \Delta$

Thus $\|f\| \leq W_n\omega_n(f.\Delta) + \|P_n(\mathrm{x})\|, x \in \Delta$

Now by theorem 3.3 we get:

$$\|f\| \leq W_n\omega_n(f.\Delta) + \|P_n(\mathrm{x})\| \leq W_n 2^n \|f\|_{c_\Delta} + \|P_n(\mathrm{x})\|, x \in \Delta$$

And then $\|f\| - W_n 2^n \|f\|_{c_\Delta} \leq \|P_n(\mathrm{x})\|$ And so $\|f\|(1 - W_n 2^n) \leq \|P_n(\mathrm{x})\|$

So, $\|f\| \leq \frac{\|P_n(\mathrm{x})\|}{(1 - W_n 2^n)}$ Not that $\|f\| \geq 0$

This mean $\|f\| \leq \frac{\|P_n(\mathrm{x})\|}{(1 - W_n 2^n)}$ if $1 - W_n 2^n \geq 0$ and $\|f\| \geq \frac{\|P_n(\mathrm{x})\|}{(1 - W_n 2^n)}$ if $1 - W_n 2^n \leq 0$

Since $1 - W_n 2^n$ is in the denominator then $\frac{1}{1 - W_n 2^n}$ is very little especially As $n \to \infty$

Certainly, this feature is available for the function f at L_p-norm.

Sure, one can assume there exist a constant $c > 0$ *or* $c \leq 0$ such that $c = 1 - W_n 2^n$

This mean $\|\mathrm{P}_n(\mathrm{x})\|, x \in I$ does not increase much on $\|f\|, x \notin I$

And then the unction $f \bigcup \mathrm{P}_n$ is a smooth function in its domain.

4. Conclusions

In this papers we concluded that it is possible to be approximate non-continuous functions by completing the set of non-continuity points through union it with an appropriate polynomial under the influence suitable norm and we also concluded a degree of the error for this approximation.

References

[1] Lubinsky DS. A survey of weighted polynomial approximation with exponential weights. Surv Approx Theor 2007;3: 1—105.

[2] Leviatan Dany, Lubinsky Doron S. The degree of shape preserving weighted polynomial approximation. USA: School of Mathematics, Georgia Institute of Technology; 2008. GA 30332-0160.

[3] Sendov Blagovest, Popove Vasil A. The averaged mouduli of smoothness. Johan Wily&Sons; 1988. p. 181.

[4] Solomon Lefschetz. Algebraic topology. American Mathematical Society Providence, Rhode Island, Colloquium Publications, Volume 27.

[5] Shadrin A, DAMTP. Approximation theory University of Cambridge, Mathematical Tripos: Part III, Lent Term. 2005.

[6] Lorentz GG. Approximation of functions. USA: Holt, Rinehart and Winston, Inc., Library of Congress catalog; 1966. 66-13296, 25323-0116.