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Abstract

Let R be a ring with identity. An R module W is called small quasi-Dedekind (small quasi-D) if for each homomorphism $h : W \to W$ such that h is nonzero, then $kerh$ is small in W. In this paper we introduce a new concept called Large-small quasi-Dedekind (L-small quasi-D) module which is a generalization of small quasi-Dedekind module.

Keywords: Small submodule, Large-small submodule, Quasi-Dedekind module, Small quasi-Dedekind module, Large-small quasi-Dedekind module

1. Introduction

Let R be a ring with identity. In [1], a submodule V of an R-module W is called small in W ($V \ll W$), if $V + S = M$, where S be a submodule of W, implies that $S = W$ [2,3]. In [5], introduce the concept of Large-small submodule as, a submodule V of an R-module W is called Large-small in W ($V \ll_L W$), if $V + S = M$, where S be a submodule of W, implies that S is essential submodule in W. Recall that An R-module W is called small quasi-Dedekind (small quasi-D), if for each nonzero homomorphism $h : W \to W$, implies $ker\, h$ is small in W [4]. In this paper we give and explain the concept of Large-small quasi-Dedekind module such that, an R-module W is called Large-small quasi-Dedekind (L-small quasi-D), if for each nonzero homomorphism $h : W \to W$, implies $ker\, h$ is Large-small in W ($ker\, h \ll_L W$).

Now, we give the main results and propositions.

Definition(1.1):
An R-module W is called L-small quasi-D, if for each nonzero homomorphism $h : W \to W$, implies that $ker\, h \ll_L W$.

***Remarks and Examples (1.2)*:**

1 Every small quasi-D is L-small quasi-D, since by [5], every small submodule is L-small. But the converse is not true, for example Z_{12} as Z-module is L-small quasi-D, since define $h : Z_{12} \to Z_{12}$ by $h(\bar{x}) = 4\bar{x}$; $\bar{x} \in Z_{12}$ so $h \neq 0$ and $ker\, h = \{\bar{x} \in Z_{12} : h(\bar{x}) = \bar{0}\} = \{\bar{x} \in Z_{12} : 4\bar{x} = \bar{0}\} = \{0, 3, 6, 9\} = 3Z_{12} \ll_L Z_{12}$ since $3Z_{12} + 2Z_{12} = Z_{12}$ and $2Z_{12}$ is essential in Z_{12}, but $3Z_{12}$ is not small in Z_{12}, and hence Z_{12} is not small quasi-D.

2 Z_4 as Z-module is L-small quasi-D. Since define $h : Z_4 \to Z_4$ by $h(\bar{x}) = 2\bar{x}$; $\bar{x} \in Z_4$ so $h \neq 0$ and $ker\, h = \{\bar{x} \in Z_4 : h(\bar{x}) = \bar{0}\} = \{\bar{x} \in Z_4 : 2\bar{x} = \bar{0}\} = \{0, 2\} \ll_L Z_4$.

3 Z_6 as Z-module is not L-small quasi-D, since let $h : Z_6 \to Z_6$, define by $h(\bar{x}) = 2\bar{x}$, $h \neq 0$ then $ker\, h = \{0, 3\}$ is not L-small in Z_6.

4 Z as Z-module is L-small quasi-D. Since define $h : Z \to Z$ by $h(\bar{x}) = 2\bar{x}$; $\bar{x} \in Z$ so $h \neq 0$ and $ker\, h = \{\bar{x} \in Z : h(\bar{x}) = \bar{0}\} = \{\bar{x} \in Z : 2\bar{x} = \bar{0}\} = \{0\} \ll_L Z$.

5 Every integral domain is L-small quasi-D but the converse is not true for example Z_4 as Z-module is L-small quasi-D but not integral domain.

6 Let W be a semisimple module, then W is small quasi-D if and only if, W is L-small quasi-D.

Proof. $(\Rightarrow)$ Clear
$(\Leftarrow)$ Let W is L-small quasi-D, then $ker\, h \ll_L W$ and since W is semisimple then $ker\, h = 0 \ll_L W$, hence $ker\, h = 0 \ll W$ so we get, W is small quasi-D.

Theorem(1.3). Let W be an R-module then W is L-small quasi-D if and only if, $Hom(\frac{W}{V}, W) = 0$ for all $V \ll_L W$.

Proof. $(\Rightarrow)$ Assume there exists V not L-small in W such that $Hom(\frac{W}{V}, W) \neq 0$, then there exists $g : \frac{W}{V} \to W$ such that $g \neq 0$ and so $go\varnothing \in End_R(W)$ where $\varnothing$ is

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canonical projection and $go\varnothing \neq 0$, hence $ker(go\varnothing) \ll_L W$ and since $V \leq go\varnothing \ll_L W$ then $V \ll_L W$ by [5] and this contradiction.
$(\Leftarrow)$ Assume there exists a nonzero homomorphism $h : W \to W$ such that $ker\ h \ll_L W$. Define $q : \frac{W}{Kerh} \to W$ by $q(w + Kerh) = h(w)$ for all $w \in W$, so clear that q is well define and $q \neq 0$ and hence, $Hom\left(\frac{W}{Kerh}, W\right) \neq 0$ and this contradiction.

Proposition(1.4). Let W be an R-module and $\overline{R} = R/U$ such that U is an ideal of R and $U \leq ann_R(W)$, then W is L-small quasi-D R-module if and only if, W is L-small quasi-D $\overline{R}$-module.

Proof. $(\Rightarrow)$ By [1], we have $Hom_R\left(\frac{W}{S}, W\right) = Hom_{\overline{R}}\left(\frac{W}{S}, W\right)$ for all $S \leq W$ and since W is L-small quasi-D R-module, then $Hom_R\left(\frac{W}{S}, W\right) = 0$ for all $S \ll_L W$ by Theorem(1.3) and hence $Hom_{\overline{R}}\left(\frac{W}{S}, W\right) = 0$ for all $S \ll_L W$, so we get W is L-small quasi-D $\overline{R}$-module.
$(\Leftarrow)$ By the same way we get the result.

Proposition(1.5). Let W_1 and W_2 be R-modules such that $W_1 \cong W_2$, then W_1 is L-small quasi-D R-module if and only if, W_2 is L-small quasi-D R-module.

proof. $(\Rightarrow)$ Let $h : W_2 \to W_2$ be a nonzero, since $W_1 \cong W_2$ then there exists an isomorphism $q : W_1 \to W_2$. By the following $W_1 \xrightarrow{q} W_2 \xrightarrow{h} W_2 \xrightarrow{q^{-1}} W_1$, we have $f = q^{-1} o\ h\ o\ q \in End_R(W_1)$, $f \neq 0$ and since W_1 is L-small quasi-D, we get $Kerf \ll_L W_1$ and so, $q(Kerf) \ll_L W_2$ by [6]. By the same proof in [4], $Kerh = q(Kerf)$, so $Kerh = q(Kerf) \ll_L W_2$ and hence $Kerh \ll_L W_2$.
$(\Leftarrow)$ By the same way we get the result.

Proposition(1.6). Let W be an R-module such that W is L-small quasi-D and quasi-injective and $V \leq W$ such that V is direct summand of W, then V is L-small quasi-D.

Proof. Let $h : V \to V$ be a nonzero, since W is quasi-injective then there exist $q : W \to W$ such that $qoi = ioh$ where i is inclusion map, and hence $q(V) = h(V) \neq 0$, also since W is L-small quasi-D so $q \neq 0$ and $Kerq \ll_L W$. Now since $Kerh \leq Kerq \ll_L W$, then $Kerh \ll_L W$ by [5] and since $Kerh \leq V$ and V is direct summand of W, we get $Kerh \ll_L V$ by [5], so V is L-small quasi-D.

Remark(1.7). Let W be an R-module and $V \leq W$, if $\frac{W}{V}$ is L-small quasi-D, then it is not necessary that W is L-small quasi-D.
For example: Let $W = Z_6$ and $= 2Z_6$, then $\frac{Z_6}{2Z_6} \cong Z_2$ that is L-small quasi-D, but Z_6 is not L-small quasi-D.

Remark(1.8). : Let W be an R-module and $V \leq W$, if W is L-small quasi-D, then it is not necessary that $\frac{W}{V}$ is L-small quasi-D.
For example: Let $W = Z$ such that Z is L-small quasi-D and let $V = 6Z$, then $\frac{Z}{6Z} \cong Z_6$ which is not L-small quasi-D.

Proposition(1.9). Let W be an R-module such that W is L-small quasi-D and let $\frac{W}{U}$ is projective for all U is not L-small in W then $\frac{W}{V}$ is L-small quasi-D for all $V \leq W$.

Proof. Suppose that $\frac{U}{V}$ is not L-small in $\frac{W}{V}$, so U is not L-small in W. Now Assume that $Hom\left(\frac{W/V}{U/V}, \frac{W}{V}\right) \neq 0$, but since $Hom\left(\frac{W/V}{U/V}, \frac{W}{V}\right) \cong Hom\left(\frac{W}{U}, \frac{W}{V}\right)$, then there exists $h : \frac{W}{U} \to \frac{W}{V}$ and $h \neq 0$, and since $\frac{W}{U}$ is projective, then there exists $q : \frac{W}{U} \to W$ such that $\varnothing oq = h$; $\varnothing$ is canonical projection, hence $\varnothing oq\left(\frac{W}{U}\right) = h\left(\frac{W}{U}\right) \neq 0$, so $q \neq 0$ but $q \in Hom\left(\frac{W}{U}, W\right)$, then $Hom\left(\frac{W}{U}, W\right) \neq 0$ and U is not L-small in W, so we get W is not L-small quasi-D and this contradiction, hence $\frac{W}{V}$ is L-small quasi-D.

Proposition(1.10). Let W be quasi-projective module and V is closed submodule in W and for each $q \in End_R(W)$ such that $q^{-1}(V) \ll_L W$, then $\frac{W}{V}$ is L-small quasi-D.

Proof. Suppose $h : \frac{W}{V} \to \frac{W}{V}$ be a nonzero, since W is quasi-projective, then there exists a homomorphism $q : W \to W$ such that $\varnothing oq = ho\varnothing$; $\varnothing$ is canonical projection. Let $Kerh = \frac{L}{V} = \{k + V : h(k + V) = V\} = \{k + V : ho\varnothing(k) = V\} = \{k + V : \varnothing oq(k) = V\} = \{k + V : q(k) + V = V\} = \{k + V : q(k) \in V\} = \{k + V : k \in q^{-1}(V)\}$, hence $Kerh = \frac{q^{-1}(V)}{V}$ and since $q^{-1}(V) \ll_L W$ and V is closed submodule in W, then $\frac{q^{-1}(V)}{V} \ll_L \frac{W}{V}$ by [5], so we get $Kerh \ll_L \frac{W}{V}$ and $\frac{W}{V}$ is L-small quasi-D.

Proposition(1.11). Let W be an R-module then W is L-small quasi-D if and only if, there exists V is L-small in W, V is fully invariant such that for each $h \in End_R(W)$, $h \neq 0$, $h(W) \not\subset V$ and $\frac{W}{V}$ is L-small quasi-D.

Proof. $(\Rightarrow)$ Choose $V = (0)$, so V is L-small in W, V is fully invariant and for each $h \in End_R(W)$, $h \neq 0$, hence $h(W) \not\subset (0) = V$ and $\frac{W}{V} = \frac{W}{(0)} \cong W$ is L-small quasi-D.
$(\Leftarrow)$ If $V = (0)$, then W is L-small quasi-D. Suppose $V \neq (0)$, such that V is L-small in W, and let $h \in End_R(W)$, $h \neq 0$, define $q : \frac{W}{V} \to \frac{W}{V}$ by

$q(w+V) = h(w) + V$ for all $w \in W$ and by the same proof in [4], q is well define and $q \neq 0$. Now since $\frac{W}{V}$ is L-small quasi-D, then $Kerq \ll_L \frac{W}{V}$, let $Kerq = \frac{L}{V} \ll_L \frac{W}{V}$ and hence $L \ll_L W$ by [5] and since $Kerh \leq L \ll_L W$, then by [5] we get $Kerh \ll_L W$ and so, W is L-small quasi-D.

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