On (K*-N)n-Quasi Normal Operators

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ARTICLE

On $(K^{*} - N)^{n}-$ Quasi Normal Operators

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Abstract

The objective is to present a novel variant of a quasi-normal operator, specifically the $(K^{*} - N)^{n}$ quasi normal operator, alongside the introduction of related theorems, propositions, and illustrative examples elucidating this concept. Additionally, we present the necessary and sufficient conditions for addition and multiplication of this kind of operators.

Keywords: Operators, Normal operators, Quasi-normal operator, K*quasi-normal operators

1. Introduction

A.Brow [1] introduced and researched the quasi-normal operator for the first time in 1953. Later Arun B. (1976) [2] provided several properties of the quasi-normal operator. Sh. Lohaj (2010) [3] introduced a novel variation of the quasi-normal operator, namely, N-quasi normal operators, along with presenting some of its fundamental characteristics. Ould A. (2011) submitted the n-power quasi-normal operator along with outlining its characteristics [4]. Valdete R.H. (2013) [5] provided several characteristics of N-quasi normal operators. Saad S. and Laith K (2015) [6] introduced the concept of the quasi-normal operator along with presenting some fundamental properties of this concept. Ahmed M and Salim D. (2017) [7] introduced another type called the $(K - N)^{*}$ quasi-normal operator, along with presenting some properties.

2. Basics

Definition (2.1). [8]
A bounded linear operator $T : \mathscr{K} \rightarrow \mathscr{K}$, is n operator satisfying $\|Tx\| \leq k\|x\|\ \forall x \in \mathscr{K}$.

Definition (2.2). [8]
A bounded linear operator $T : \mathscr{K} \rightarrow \mathscr{K}$, is called normal if $TT^{*} = T^{*}T$.

Definition (2.3). [1]
A bounded linear operator $T : \mathscr{K} \rightarrow \mathscr{K}$ is quasi-normal if $T(T^{*}T) = (T^{*}T)T$.

Theorem (2.4). [2]
Le t $T_1, T_2 : \mathscr{K} \rightarrow \mathscr{K}$ be quasi normal operators, such that $T_2T_1 = T_1T_2 = {T_1}^{*}T_2 = T_2{T_1}^{*} = 0$ then $T_1 + T_2$ is quasi normal.

Definition (2.5). [5]
A bounded linear operator $T : \mathscr{K} \rightarrow \mathscr{K}$ be a bounded linear is N-quasi normal if $T(T^{*}T) = N(T^{*}T)T$.

Theorem (2.6). [5]
Let $T_1 : \mathscr{K} \rightarrow \mathscr{K}$ be N-quasi normal and $T_2 : \mathscr{K} \rightarrow \mathscr{K}$ be quasi normal then the product T_1T_2 is N-quasi normal if the they satisfy $T_1T_2 = T_2T_1$ and ${T_2}^{*}T_1 = T_1T_2^{*}$.

Definition (2.7). [4]
A bounded liner operator $T : \mathscr{K} \rightarrow \mathscr{K}$ is n-power quasi-normal if $T^{n}(T^{*}T) = (T^{*}T)T^{n}$

Theorem (2.7). [9]
If $T : \mathscr{K} \rightarrow \mathscr{K}$ is n-power quasi-normal operator, then λT is also a n-power quasi-normal operator, where $\lambda \in R$.

Definition (2.8). [10]
Let $T : \mathscr{K} \rightarrow \mathscr{K}$ is a bounded liner operator is said to be (K-N) quasi-normal operator if $T^{k}(T^{*}T) = N(T^{*}T)T^{k}$.

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Theorem (2.9). [10]
Let T_1 be (K-N) quasi normal operator and T_2 is k-quasi normal operator, such that, $(T_1T_2) = (T_2T_1)$ and $T_1T_2^* = T_2^*T_1$, then T_1T_2 is (K-N) quasi normal operator.

Theorem (2.10). [10]
Let $T_1, T_2 : \mathscr{K} \to \mathscr{K}$ be two (K-N) quasi normal operators such that $T_1^kT_2^* = T_2^kT_1^* = T_1^*T_2 = T_2^*T_1 = 0$ then $T_1 + T_2$ is (k-N) quasi normal operator.

Corollary (2.11). [10]
Let $T_1, T_2 : \mathscr{K} \to \mathscr{K}$ be two (K-N) quasi normal operators, such that $T_1^kT_2^* = T_2^kT_1^* = T_1^*T_2 = T_2^*T_1 = 0$ then $T_1 - T_2$ is (K-N) quasi normal operator.

3. Main results

Definition (3.1). Let $T : \mathscr{K} \to \mathscr{K}$ be a bounded linear operator then T is said to be $(K^* - N)^n$ quasi normal operator ($(K^* - N)^n$-QNO) if $(T^*)^K(T^*T)^n = N(T^*T)^n(T^*)^K$.

Example (3.2). $T = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$ is $(K^* - N)^n$-QNO where $k = 1, n = 1, N = I$

Propositions (3.3). Let $T : \mathscr{K} \to \mathscr{K}$ is $(K^* - N)^n$-QNO then λT is $(K^* - N)^n$-QNO where $\lambda \in R$.

Proof.

$$((\lambda T)^*)^K((\lambda T)^*(\lambda T))^n = (\lambda T^*)^K((\lambda T^*)(\lambda T))^n$$

$$= \lambda^K(T^*)^K(\lambda T^*)^n(\lambda T)^n$$

$$= \lambda^K(T^*)^K\lambda^n(T^*)^n\lambda^nT^n$$

$$= \lambda^{K+2n}(T^*)^K(T^*)^nT^n$$

$$= \lambda^{K+2n}N(T^*)^nT^n(T^*)^K$$

$$= N\lambda^n(T^*)^n\lambda^nT^n\lambda^K(T^*)^K$$

$$= N(\lambda T^*)^n(\lambda T)^n(\lambda T^*)^K$$

$$= N((\lambda T)^*(\lambda T))^n((\lambda T)^*)^K$$

$((\lambda T)^*)^K(\lambda T)^*(\lambda T)^n = N(\lambda T)^*(\lambda T)^n((\lambda T)^*)^K$, hence the λT is $(K^* - N)^n$ − QNO ■.

Propositions (3.4). Let $T : \mathscr{K} \to \mathscr{K}$ is $(K^* - N)^n$-QNO then T/M is $(K^* - N)^n$ -QNO where M is closed sub space.

$$\left(\left(T/M\right)^*\right)^K\left(\left(T/M\right)^*\left(T/M\right)\right)^n$$

$$= \left(\left(T^*/M\right)\right)^K\left(\left(T^*/M\right)\left(T/M\right)\right)^n$$

$$= \left((T^*)^K/M\right)\left(T^*T/M\right)^n$$

$$= \left((T^*)^K/M\right)\left((T^*T)^n/M\right)$$

$$= \left(N(T^*T)^n(T^*)^K/M\right)$$

$$= \left(N(T^*T)^n/M\right)\left((T^*)^K/M\right)$$

$$= \left(N\left(\left(T^*/M\right)\left(T/M\right)\right)^n\right)\left(\left(T^*/M\right)\right)^K$$

$$= N\left(\left(T/M\right)^*\left(T/M\right)\right)^n\left(\left(T/M\right)^*\right)^K$$

Then T/M is $(K^* - N)^n$-QNO ■

Remark (3.5). Let T_1 and T_2 be two $(K^* - N)^n$-QNO then $T_1 + T_2$ is not necessary $(K^* - N)^n$-QNO. To illustrate that consider the following example.

Example (3.6).

$$T_1 = \begin{bmatrix} 2i & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -2i \end{bmatrix} \text{ and } T_2 = \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ i & 0 & 1 \end{bmatrix}$$

$$T_1 + T_2 = \begin{bmatrix} 1+2i & 0 & 2+i \\ 0 & 1 & 0 \\ 2 & 0 & -2i \end{bmatrix}$$

If $n = 1, K = 2$ and $N = I$.

$$((T_1 + T_2)^*)^2((T_1 + T_2)^*(T_1 + T_2))^1 = \begin{bmatrix} 32-56i & 0 & -24-52i \\ 0 & 1 & 0 \\ 40-20i & 0 & -40i \end{bmatrix} \quad (1)$$

$$((T_1+T_2)^*(T_1+T_2))^1((T_1+T_2)^*)^2 = \begin{bmatrix} -120i & 0 & 40-20i \\ 0 & 1 & 0 \\ 104-52i & 0 & 32+24i \end{bmatrix} \quad (2)$$

$(1) \neq (2)$

Then T_1+T_2 is not $(K^*-N)^n$-QNO

Theorem (3.7). Let $T_1, T_2 : \mathscr{K} \to \mathscr{K}$ be $(K^*-N)^n$-QNO on a Hilbert space s.t. $T_1^*T_2^* = T_1T_2 = T_1^*T_2 = 0$ then $T_1+T_2(K^*-N)^n$-QNO.

Proof. -

$$\left(((T_1+T_2)^*)^K((T_1+T_2)^*(T_1+T_2))^n\right)$$

$$= \left((T_1^*+T_2^*)^K((T_1^*+T_2^*)(T_1+T_2))^n\right)$$

$$= \left((T_1^*)^K + K(T_1^*)^{K-1}T_2^* + \ldots + (T_2^*)^K\right)\left(\left(T_1^* + n(T_1^*)^{n-1}T_2^* + \ldots + (T_2^*)^n\right)(T_1^n + nT_1^{n-1}T_2 + \ldots + T_2^n)\right) = \left((T_1^*)^K + (T_2^*)^K\right)(((T_1^*)^n + (T_2^*)^n)(T_1^n + T_2^n))$$

$$= \left((T_1^*)^K + (T_2^*)^K\right) \times ((T_1^*)^nT_1^n + (T_1^*)^nT_2^n + (T_2^*)^nT_1^n + (T_2^*)^nT_2^n)$$

$$= \left((T_1^*)^K + (T_2^*)^K\right)((T_1^*)^nT_1^n + (T_2^*)^nT_2^n)$$

$$= \left((T_1^*)^K(T_1^*)^nT_1^n + (T_1^*)^K(T_2^*)^nT_2^n + (T_2^*)^K(T_1^*)^nT_1^n + (T_2^*)^K(T_2^*)^nT_2^n\right)$$

$$= \left((T_1^*)^KT_1^*T_1^n + (T_2^*)^KT_2^*T_2^n\right)$$

Since T_1 and T_2 are $(K^*-N)^n$-QNO

$$= N\left((T_1^*)^nT_1^n(T_1^*)^K\right) + N\left((T_2^*)^nT_2^n(T_2^*)^K\right)$$

$$= N\left(\left((T_1^*)^nT_1^n(T_1^*)^K\right) + \left((T_2^*)^nT_2^n(T_2^*)^K\right)\right)$$

Therefore; T_1+T_2 is $(K^*-N)^n$-QNO ■

Remark (3.8). Let T_1 and T_2 be two $(K^*-N)^n$-QNOs then T_1T_2 is not necessarily a $(K^*-N)^n$-QNO as shown below.

Example (3.9).

$$T_1 = \begin{bmatrix} 2i & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -2i \end{bmatrix} \text{ and } T_2 = \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ i & 0 & 1 \end{bmatrix}$$

$$T_1T_2 = \begin{bmatrix} 4i & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix}$$

If $n = 1, K = 2$ and $N = I$.

$$((T_1T_2)^*)^2((T_1T_2)^*(T_1T_2))^1 = \begin{bmatrix} -512 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ldots(1)$$

$$((T_1T_2)^*(T_1T_2))^1((T_1T_2)^*)^2 = \begin{bmatrix} -512 & 0 & -512i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ldots(2)$$

$(1) \neq (2)$

Then T_1T_2 is not $(K^*-N)^n$-QNO

Theorem (3.10). Let T_1 *and* $T_2 : \mathscr{K} \to \mathscr{K}$ be $(K^*-N)^n$-QNO and K^* quasi normal operator respectively such that, $T_1T_2 = T_2T_1$ and ${T_2}^*T_1 = T_1T_2^*$, then T_1T_2 is $(K^*-N)^n$ -QNOs.

Proof.

$$((T_1T_2)^*)^K((T_1T_2)^*(T_1T_2))^n = ((T_2T_1)^*)^K\left((T_2^*T_1^*)^n(T_1T_2)^n\right)$$

$$= \left((T_1^*T_2^*)\right)^K\left((T_1^*T_2^*)^n(T_1T_2)^n\right)$$

$$= \left((T_1^*)^K(T_2^*)^K\right)(((T_1^*)^n(T_2^*)^n)(T_1^nT_2^n))$$

$$= (T_1^*)^K\left((T_2^*)^K(T_1^*)^n\right)((T_2^*)^nT_1^n)T_2^n$$

$$= (T_1^*)^K\left((T_1^*)^n(T_2^*)^K\right)(T_1^n(T_2^*)^n)T_2^n$$

$$= \left((T_1^*)^K(T_1^*)^n\right)\left((T_2^*)^KT_1^n\right)((T_2^*)^nT_2^n)$$

$$= \left((T_1^*)^K(T_1^*)^n\right)\left(T_1^n(T_2^*)^K\right)((T_2^*)^nT_2^n)$$

$$= \left((T_1^*)^K(T_1^*)^nT_1^n\right)\left((T_2^*)^K(T_2^*)^nT_2^n\right)$$

$$= N\left(T_1^*T_1^n(T_1^*)^K\right)\left((T_1^*)^nT_2^n(T_2^*)^K\right)$$

$$= N((T_1^*)^nT_1^n)\left((T_1^*)^K(T_2^*)^n\right)\left(T_2^n(T_2^*)^K\right)$$

$$= N((T_1^*)^nT_1^n)\left((T_2^*)^n(T_1^*)^K\right)\left(T_2^n(T_2^*)^K\right)$$

$$= N(T_1^*)^n(T_1^n(T_2^*)^n)\left((T_1^*)^KT_2^n\right)(T_2^*)^K$$

$$=N\left(T_1^*\right)^n\left(\left(T_2^*\right)^n T_1^n\right)\left(T_2^n\left(T_1^*\right)^K\right)\left(T_2^*\right)^K$$

$$=N\left(T_1^*T_2^*\right)^n\left(T_1^nT_2^n\right)\left(\left(T_1^*\right)^K\left(T_2^*\right)^K\right)$$

$$=N((T_2T_1)^*)^n(T_1T_2)^n\left(\left(T_1^*\right)\left(T_2^*\right)\right)^K$$

$$=N((T_2T_1)^*)^n(T_1T_2)^n((T_2T_1)^*)^K$$

$$=N((T_1T_2)^*)^n(T_1T_2)^n((T_1T_2)^*)^K$$

$$((T_1T_2)^*)^K((T_1T_2)^*)^n(T_1T_2)^n =N((T_1T_2)^*)^n(T_1T_2)^n((T_1T_2)^*)^K$$

Therefore; the product T_1T_2 is $(K^* - N)^n$-QNO ■

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