Al-Qadisiyah Journal of Pure Science

Manuscript 1265

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ARTICLE

Using a Combination Technique of Two Computational Methods to Solve the Integro-differential Equation

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Abstract

The present paper investigates a mathematical technique for determining nonlinear integro-differential equations that is based on a combination of the Bownds and Range-Kutta methods. There will be a discussion of what is meant by this class of nonlinear Fredholm equations, as well as an explanation of the mathematical method for this kind of equation. The Bownds technique is used to produce the differential equations from the integral equation obtained from the integro-differential equations. Then, the approximate solution of the differential equation(s) is derived by applying the Range-Kutta method. Through solving three different examples, we prove that the combination of mathematical methods is efficient and reliable. This makes it a valuable tool for determining nonlinear Fredholm equations with different degenerate kernels quickly and accurately. Finally, it is shown that the importance of my research is that it contributes to providing modern strategies for the numerical solution of this type of equation and it has the potential to lead to more efficient numerical methods for solving nonlinear equations.

Keywords: Bownds method, Runge-Kutta method of fourth order, Nonlinear integro-differential equation of Fredholm type, Approximate solutions, Degenerate kernel, Initial value problem, System of first order differential equations

1. Introduction

A mathematical technique is examined to be applied in determining the nonlinear integro-differential equation (NIDE) of Fredholm type which has degenerate kernel. This technique is known as a Bownds method (BM), it admits an approximate solution (AS) while making use of another method. The idea of the BM is to convert the degenerate kernel of integral equation to a finite series, this process products a system of first order differential equations (DEs) which can be numerically determined by employing a numerical method. Altogether, BM combined with Runge-Kutta (BRK) method are the effective and simple approach to solve integro-differential equation (IDEs).

The BM has been described and applied to solve nonlinear Volterra integral equation in [1–3]. It is also used for evaluating the solution of linear Fredholm IDEs [4].

As a numerical method, Runge-Kutta (RK) is used to determine DEs [5–7]. It can handle higher order problems and is more robust than other methods. With the RK fourth order (RK4) method, we are able to solve DEs of first order with initial conditions with high precision. As a result of using the solution of the DEs, we are able to have the solution of the NIDE [3].

The integral equations of Fredholm type arise in many scientific applications [8,9]. They can be solved using a variety of numerical methods [10–13].

The goal of our article is to investigate and examined the BRK technique for NIDEs, this will be

Received 20 March 2024; accepted 11 April 2024.
Available online 28 September 2024

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https://doi.org/10.29350/2411-3514.1265

illustrated by determining three different examples based on the mathematical method that will be introduced in the next section.

The paper is organized in the following way: Section 2 presents the BRK method including the steps of technique for determining the NIDE of Fredholm type. In Section 3, three different examples of the NIDE, will be evaluated by applying the BRK method, the results of the solutions will be introduced using tables and figures. Section 4 will be allocated for the conclusions of investigation of the BRK method.

2. Mathematical method

BM requires the conversion of the IDE with degenerate kernel to the integral equation. After converting the degenerate kernel into an appropriate finite sum, BM converts the integral equation into a first order DEs. The next step is to use the RK method of fourth order to find the solution to the problem. This structure of method can be illustrated by the following steps:

Let us first introduce the most standard form of the NIDE of order (n) as follows:

$$\mathscr{P}^{(n)}(r)=S(r)+\int_{H(r)}^{G(r)}(r-v)^n\mathscr{N}(\mathscr{P}(v))dv, \tag{1}$$

with initial conditions $\mathscr{P}(0)=\alpha, \mathscr{P}'(0)=\beta,\ldots,\mathscr{P}^{(n-1)}(0)=\gamma,$ for the determination of the particular solution $\mathscr{P}(r)$, in eq. (1), $\mathscr{P}(r)$ is a continuous function and the n^{th} derivative of the unknown function $\mathscr{P}(r)$ that will be evaluated, n is an integer number, $n\geq 0$, $S(r)$ is known analytic function, $(r-v)^n$ is the a known function of two variable r and v that is called kernel of IDE, the kernel function in this paper will be taken as degenerate type function, $H(r)$ and $G(r)$ are limits of integration that may be both constants, variables or mixed, and $\mathscr{N}(\mathscr{P}(v))$ is a nonlinear unknown function.

Now, in our study, we will consider the class of Fredholm NIDE. That means, the limits in eq. (1) will be constants c and g,

$$\mathscr{P}^{(n)}(r)=S(r)+\int_{c}^{g}(r-v)^n\mathscr{N}(\mathscr{P}(v))dv, \tag{2}$$

Now, integrate both sides of (2) n^{th} times with respect to r, and utilizing the initial conditions $\mathscr{P}^{(s)}(0), s=0,1,\ldots,n-1$, we get

$$\mathscr{P}(r)=\mathscr{F}(r)+\int_{c}^{g}\mathscr{K}(r,v)\mathscr{N}(\mathscr{P}(v))dv, \tag{3}$$

where,

$$\mathscr{K}(r,v)=\frac{1}{(n+1)!}(r-v)^{n+1}, \text{and}$$
$$\mathscr{F}(r)=\mathscr{P}(c)+r\mathscr{P}'(c)+\ldots+\frac{r^{n-1}}{(n-1)!}\mathscr{P}^{(n-1)}(c).$$

We can say that, $\tilde{\mathscr{K}}(r,v,\mathscr{P}(v))=\mathscr{K}(r,v)\mathscr{N}(\mathscr{P}(v))$ can be expressed by writing a finite series:

$$\tilde{\mathscr{K}}(r,v,\mathscr{P}(v))=\sum_{i=1}^{M}A_i(r)\,B_i(v,\mathscr{P}(v)), \tag{4}$$

here, A_i and B_i are two continuous functions such that $A_i:[c,g]\rightarrow\mathbb{R}^n$, $B_i:[c,g]\times\mathbb{R}^n\rightarrow\mathbb{R}$, $c\leq v\leq r\leq g$ and $|\mathscr{P}|<+\infty$, then, the solution of eq. (3) can be expressed in the form:

$$\mathscr{P}(r)=\mathscr{F}(r)+\sum_{i=1}^{M}A_i(r)\,Z_i(r), \tag{5}$$

where,

$$Z_i'(r)=B_i\left(r,\mathscr{F}(r)+\sum_{i=1}^{M}A_i(r)\,Z_i(r)\right),Z_i(c)=0,\quad i=1,2,\ldots,M,M\in\mathbb{Z}^+. \tag{6}$$

Here, eq. (6) is a system of first order DEs with initial conditions, this can be solved by applying the RK method to find $Z_i(r)$ and then, substitute it in eq. (5) to find the solution of (1).

3. Numerical applications

The BRK method is applied here to three different examples of NIDE. The results will be shown in tables comparing ASs with exact solutions (ESs), and the efficiency of these solutions will also be explained in figures.

Example 3.1. Take into consideration the Fredholm NIDE:

$$\mathscr{P}''(r)=S(r)+\int_{0}^{1}(r-v)\,\mathscr{P}^2(v)\,dv, \tag{7}$$

where,

$$S(r)=9e^r-\frac{7}{2}r-\frac{31}{2}-\frac{1}{3}r^4-\frac{1}{4}e^{2r}-4re^r,$$

and the initial values are $\mathscr{P}'(0) = 1$ and $\mathscr{P}(r) = 1$. Now, we start the BM by integrating (7) twice with respect to r, we obtain:

$$\mathscr{P}(r)=\mathscr{F}(r)+\frac{1}{6}\int_0^1 (r-v)^3\,\mathscr{P}^2(v)\,dv,$$

where,

$$F(r)=17\,e^{r}-\frac{79}{8}r-\frac{31}{8}r^2-\frac{7}{12}r^3-\frac{1}{90}r^6-4\,r\,e^{r}-\frac{1}{16}e^{2r}-\frac{255}{16},$$

Table 1. *The AS of using BKR method and compare it with the ES.*

r	ES	AS	Error
0.0	0.0000000000	0.0000000000	0.00000000e+00
0.1	0.1000000000	0.1000000020	1.99559684e-09
0.2	0.2000000000	0.2000000064	6.42253298e-09
0.3	0.3000000000	0.3000000123	1.23166577e-08
0.4	0.4000000000	0.4000000189	1.88543604e-08
0.5	0.5000000000	0.5000000254	2.53729953e-08
0.6	0.6000000000	0.6000000314	3.13505671e-08
0.7	0.7000000000	0.7000000364	3.63517595e-08
0.8	0.8000000000	0.8000000399	3.99494737e-08
0.9	0.9000000000	0.9000000416	4.16308090e-08
1.0	1.0000000000	1.0000000407	4.06942260e-08
L.S.E			8.4856e-15

and,

$$k(r,v)=\frac{1}{6}\left(r^3-3r^2+3r-1\right).$$

According to (4), we have:

$$\sum_{i=1}^{4}A_i(r)B_i(v,\mathscr{P}(v))=\frac{1}{6}\left(r^3-3r^2+3r-1\right)\mathscr{P}^3(v).$$

Then, the solution of (7) can be expressed as:

$$\mathscr{P}(r)=\mathscr{F}(r)+\sum_{i=1}^{4}A_i(r)\,Z_i(r), \tag{8}$$

where,

$$Z_i'(r)=B_i\left(r,\mathscr{F}(r)+\sum_{i=1}^{4}A_i(r)\;Z_i(r)\right),Z_i(0)=0,\quad i=1,2,\ldots,4. \tag{9}$$

Next, the RK method can be applied to find the AS of (9). Finally, using this solution $Z_i(r)$ in (8) gives the solution of NIDE (7), the results is shown in Table 1 and Fig. 1.

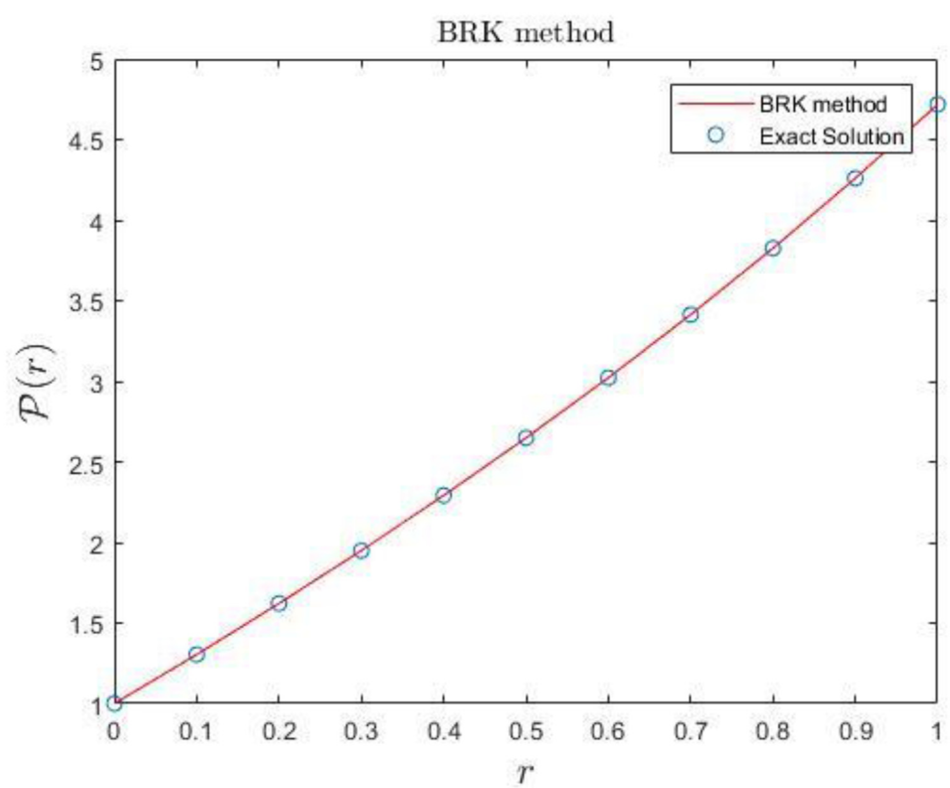

Fig. 1. The accuracy of using the BRK method, the straight line is the output of the points of $0<r<1$ by the BRK method and the circle line is the points of $0<r<1$ by the ES.

Example 3.2. Take into consideration the Fredholm nonlinear NIDE:

$$\mathscr{P}'(r) = S(r) + \int_0^1 v \sin(\mathscr{P}(v))\, dv \tag{10}$$

where,

$S(r) = 1 - r\cos(x) - \sin(x)$

and the initial values are $\mathscr{P}(0) = 0$. Now, we start the BM by integrating (6) one time with respect to r, we get:

Table 2. The AS of using BKR method and compare it with the ES.

r	ES	AS	Error
0.0	1.0000000000	1.0000000000	0.00000000e+00
0.1	1.3051708419	1.3051709181	7.62025472e-08
0.2	1.6214026182	1.6214027582	1.39945313e-07
0.3	1.9498586147	1.9498588076	1.92830853e-07
0.4	2.2918244609	2.2918246976	2.36712636e-07
0.5	2.6487209969	2.6487212707	2.73809858e-07
0.6	3.0221184935	3.0221188004	3.06865422e-07
0.7	3.4137523681	3.4137527075	3.39359672e-07
0.8	3.8255405527	3.8255409285	3.75796056e-07
0.9	4.2596026891	4.2596031112	4.22080062e-07
1.0	4.7182813424	4.7182818285	4.86018934e-07
L.S.E			9.585e-13

$$\mathscr{P}(r) = \mathscr{F}(r) + \int_0^1 v\,(r - v)\sin(\mathscr{P}(v))\, dv, \tag{11}$$

where, $F(r) = r + 2\cos(r)$-$r\sin(r)$, and $k(r, v) = v\,(r - v)$. According to (4), we have:

$$\sum_{i=1}^{2} A_i(r) B_i(v, P(v)) = v\,(r\text{-}v)\sin(P(v))$$

Then, the solution of (11) can be expressed as:

$$\mathscr{P}(r) = \mathscr{F}(r) + \sum_{i=1}^{2} A_i(r)\, Z_i(r),$$

where,

$$Z_i'(r) = B_i\left(r, \mathscr{F}(r) + \sum_{i=1}^{2} A_i(r)\, Z_i(r)\right), Z_i(0) = 0, i = 1, 2, \tag{13}$$

Using the RK method to find the AS, the result as in Table 2 and Fig. 2.

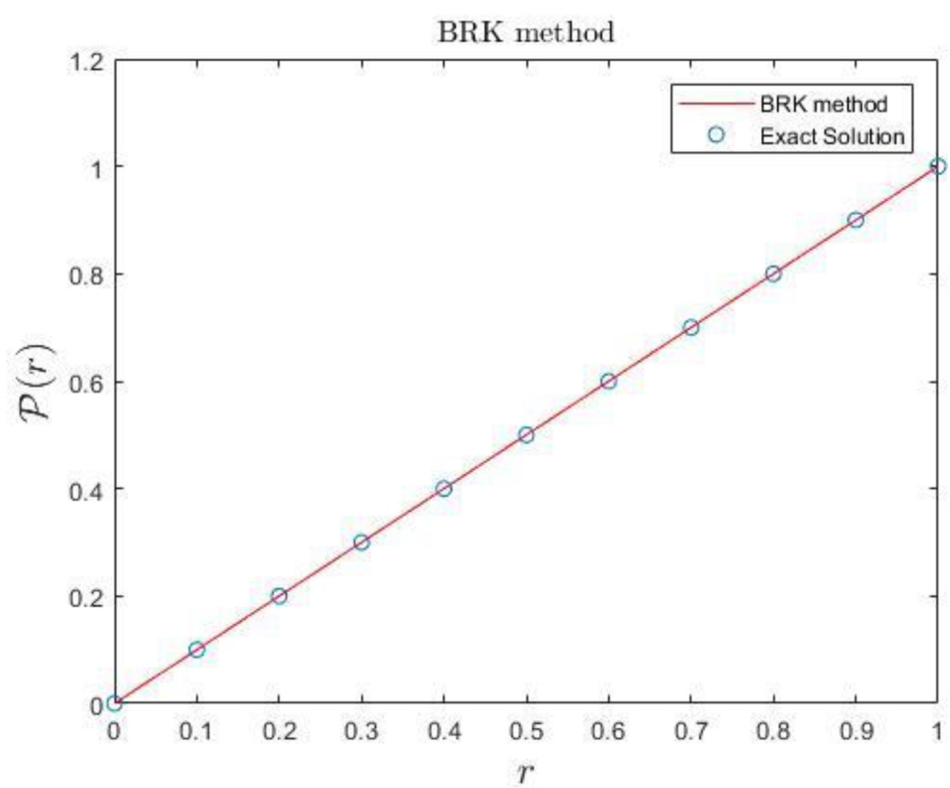

Fig. 2. The accuracy of using the BRK method, the straight line is the output of the points of $0 < r < 1$ by the BRK method and the circle line is the points of $0 < r < 1$ by the ES.

Example 3.3. Take into consideration the Fredholm nonlinear NIDE:

$$\mathscr{P}'(r) = S(r) + \int_0^1 \mathscr{P}^2(v)\, dv \tag{14}$$

where, $S(r) = 2r + 3 - \frac{1}{5}r^5 - \frac{3}{2}r^4 - 3r^3$, and the initial values are $\mathscr{P}(0) = 0$.

Now, we start the BM by integrating (6) one time with respect to r to obtain:

Table 3. The AS of using BKR method and compare it with the ES.

x	ES	AS	Error
0.0	0.0000000000	0.0000000000	0.00000000e+00
0.1	0.3100000000	0.3099994826	5.17361391e-07
0.2	0.6400000000	0.6399988245	1.17545011e-06
0.3	0.9900000000	0.9899979046	2.09544548e-06
0.4	1.3600000000	1.3599965447	3.45527900e-06
0.5	1.7500000000	1.7499944758	5.52420609e-06
0.6	2.1600000000	2.1599912740	8.72599126e-06
0.7	2.5900000000	2.5899862536	1.37464113e-05
0.8	3.0400000000	3.0399782910	2.17090187e-05
0.9	3.5100000000	3.5099655435	3.44564892e-05
1.0	4.0000000000	3.9999450033	5.49967254e-05
L.S. E			4.9968e-09

$$\mathscr{P}(r) = \mathscr{F}(r) + \int_0^1 (r - v)\, \mathscr{P}^2(v)\, dv, \tag{15}$$

where,

$$F(r) = r^2 + 3r - \frac{1}{30}r^6 - \frac{3}{10}r^5 - \frac{3}{4}r^4.$$

According to (4), we have:

$$\sum_{i=1}^{2} A_i(r) B_i(v, \mathscr{P}(v)) = (r - v)\, \mathscr{P}^2(v)$$

Then, the solution of (10) can be expressed as:

$$\mathscr{P}(r) = \mathscr{F}(r) + \sum_{i=1}^{2} A_i(r)\, Z_i(r), \tag{16}$$

where,

$$Z_i'(r) = B_i\left(r, \mathscr{F}(r) + \sum_{i=1}^{2} A_i(r)\, Z_i(r)\right), Z_i(0) = 0, i = 1, 2. \tag{17}$$

Using the RK method to find the AS, the result as in Table 3 and Fig. 3.

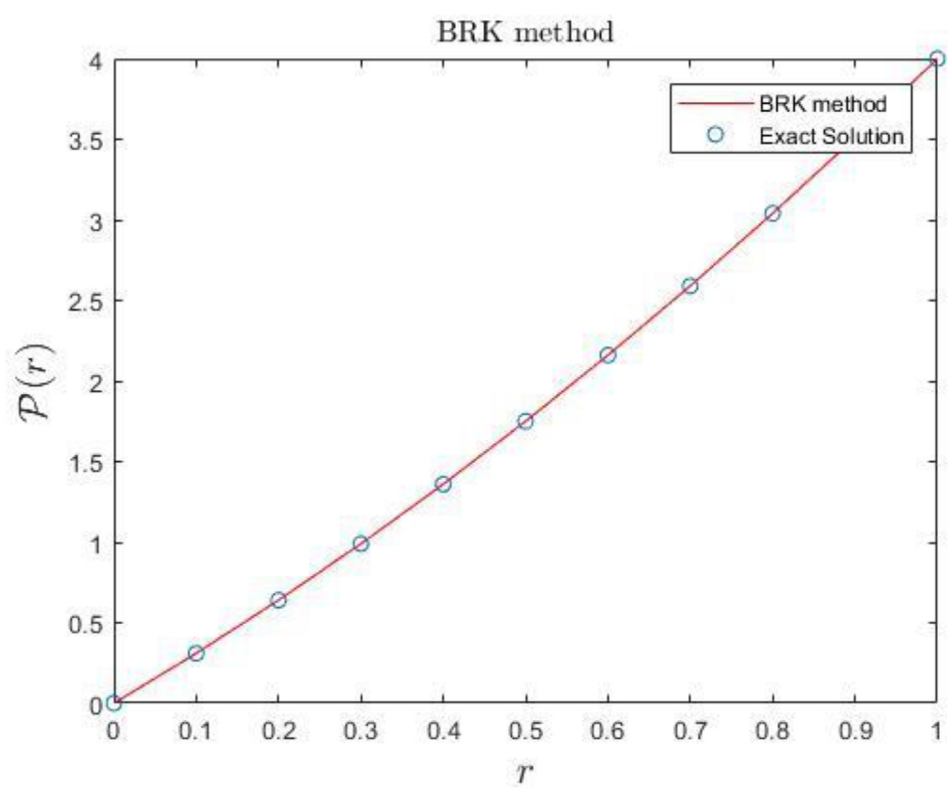

Fig. 3. The accuracy of using the BRK method, the straight line is the output of the points of $0 < r < 1$ by the BRK method and the circle line is the points of $0 < r < 1$ by the ES.

4. Conclusions

The nonlinear Fredholm IDE has been determined based on using two kinds of mathematical methods, namely, the BM, and the RK4 method. By using the first one, an integral equation has been converted into a system of first order DEs. Then, the second one, has been employed to evaluate the solution of DEs.

The mathematical method has been explained for NIDE of a nonlinear Fredholm type, involving the construction of the finite series of $\tilde{\mathscr{K}}(r, v, \mathscr{P}(v))$ then, applying the RK method to product the solution of DEs $Z_i(r)$, which can be used in (5) to find the solution of NIDE.

It is important to consider that the BM has been applied to the NIDE which had a degenerate kernel, the second point is that we employed the BM with the RK4 method to reach the solution.

The mathematical method has been examined by three examples of NIDE. As a result of comparing our AS with ES, we have noticed that our AS were reliable and very good solutions (see tables and Figs. 1–3). Finally, the results indicated and proved the ability and efficiency of BRK approach for finding the NIDEs's solutions making it a reliable technique for research.

Funding

This work is self-funded.

Acknowledgments

It is with great gratitude that the author would like to thank "College of Education for pure sciences at the University of Mosul, Iraq" for its support him and for providing the opportunity for his research.

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